The Place of Representation and Structure in Supporting Closing Gaps

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Improvement Advisers for Primary Maths

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Manipulatives and Representations

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Definitions

Manipulatives- objects that children or practitioners can interact with and move to represent mathematical ideas. *Can include everyday objects such as pine cones, buttons, and small toys as well as resources like interlocking cubes, Cuisenaire rods, Dienes blocks, and building blocks.*

Representations- a particular form in which mathematics is presented. *Can include informal drawings, mathematical symbols, and more formal diagrams, such as a number line or graphs.*

Manipulatives and Representations

- Powerful **tools for supporting** young children **engage with ideas across areas of mathematics.**
- Help children **make sense of mathematical concepts,** develop visual images, increase engagement and enjoyment.
- Help practitioners see **what children understand** and provide a **bridge to abstract thinking**.
- Children benefit from practical, first hand experiences of moving and interacting with manipulatives to develop mathematical ideas.

The use of Manipulatives and Representations

- Important that children have opportunities to engage in both **free** and **structured** play with manipulatives.
- Practitioners must help children to **understand the links** between the **manipulatives or representations** and the **mathematical ideas they represent.**
- As children's understanding of mathematical ideas develops, practitioners should **encourage children to use pictures, symbols and more abstract diagrams** to represent and communicate ideas and concepts.

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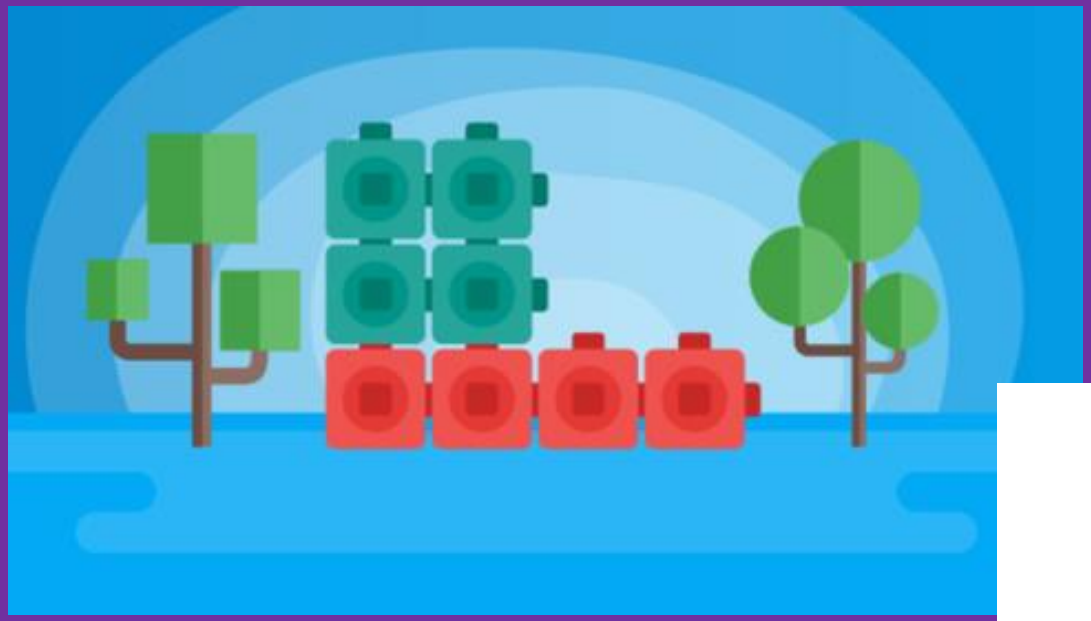
Why Manipulatives

- Evidence suggests that **physical whole-body movements** and gestures support the learning of mathematics.
- An example could be **moving along a physical number line** or **jumping and clapping while counting**.
- In the Early Years, children's use of fingers should be encouraged.
- Fingers can be useful for supporting counting and later on for counting in groups and can be important manipulatives for children.

Education Endowment Foundation



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Education Endowment Foundation Maths Report

The Education Endowment Foundation created a Maths report which researched why primary schools must continue to focus on closing the attainment gap in maths.

This has been broken into 2 documents.

- **Improving Mathematics in the Early Years and Key Stage 1**

and

- **Improving Mathematics in Key Stages 2 and 3.**

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Improving Mathematics in the Early Years and Key Stage 1

5 key recommendations from the research.

- Develop practitioners' understanding of how children learn mathematics
- Dedicate time for children to learn mathematics and integrate mathematics throughout the day
- **Use manipulatives and representations to develop understanding**
- Ensure that teaching builds on what children already know
- Use high quality targeted support to help all children learn mathematics.

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Improving Mathematics in Key Stages 2 and 3

8 key recommendations from the research.

- Use assessment to build on pupils' existing knowledge and understanding
- **Use manipulatives and representations**
- Teach strategies for solving problems
- Enable pupils to develop a rich network of mathematical knowledge
- Develop pupils' independence and motivation
- Use tasks and resources to challenge and support pupils' mathematics
- Use structured interventions to provide additional support
- Support pupils to make a successful transition between Primary and Secondary school.

Using Manipulatives and Representations to Develop Understanding- KS1

- Manipulatives and representations can be **powerful tools for supporting** young children to engage with mathematical ideas.
- Ensure that children **understand the links** between the manipulatives and the mathematical ideas they represent.
- Ensure that there is a **clear rationale** for using a particular manipulative or representation to teach a specific mathematical concept.
- Encourage children to **represent problems in their own way**, for example with drawings and marks.
- Use manipulatives and representations to **encourage discussion** about mathematics.
- Encourage children **to use their fingers—** an important manipulative for children.

Using Manipulatives and Representations – KS2

- Manipulatives (physical objects used to teach maths) and representations (such as number lines and graphs) can help pupils **engage with mathematical ideas.**
- However, **manipulatives and representations are just tools**: how they are used is essential.
- They need to be **used purposefully and appropriately** to have an impact.
- There must be a **clear rationale** for using a particular manipulative or representation to teach a specific mathematical concept.
- **Manipulatives should be temporary**; they should act as a 'scaffold' that can be removed once independence is achieved.

Effective use of Manipulatives and Representations

Understand the links between the manipulatives and the mathematical ideas they represent.

Support children in linking a manipulative with the mathematical ideas it represents.

A child may be confident using Dienes blocks to add but be unable to connect this to a written addition.

Need to **explicitly help** children **link** the materials (and the actions performed on or with them) to the mathematics of the situation.

This supports children to develop related mathematical images, representations, and symbols.

Effective use of Manipulatives and Representations

A clear rationale

What is the rationale for using a particular manipulative or representation to teach a specific mathematical concept?

Consider carefully how the manipulative will be **used to build on existing understanding**, and help develop increasingly sophisticated approaches and ideas.

Effective use of Manipulatives and Representations

Children to represent problems in their own way.

Support children to become familiar with a **repertoire of strategies** to use to represent mathematical ideas.

This could include using their fingers, drawings, and marks such as tallies and arrows.

Children should be free to **invent and explore their own representations** to record their thinking and communicate their understanding.

Effective use of Manipulatives and Representations

An awareness that children can be distracted by some manipulatives.

The surface features of a novelty manipulative can take away from the intended learning aim.

Using a given manipulative regularly, or **introducing it through play** to gain familiarity can be beneficial.

Effective use of Manipulatives and Representations

Encourage discussion about mathematics by using manipulatives and representations.

Children work in **pairs and small groups** using manipulatives to solve problems and to encourage questions about strategies and reasoning.

Prompt the **sharing and comparison of different approaches**.

Manipulatives can be used to communicate what children know.

Effective use of Manipulatives and Representations

Ensure pupils understand the links between the manipulatives and the mathematical ideas they represent.

Teachers should encourage pupils to **link the materials to the mathematics of the situation**, to appreciate the limitations of concrete materials, and to develop related mathematical images, representations and symbols.

Effective use of Manipulatives and Representations

Teachers should avoid pupils becoming reliant on manipulatives to do a type of task or question.

Manipulatives should enable pupils to understand mathematics by **illuminating the underlying general relationships.**

Manipulatives should **not just 'get pupils to the right answer'** to a specific problem.

Effective use of Manipulatives and Representations

Manipulatives should act as a ‘scaffold’ which will be removed once independence is achieved.

Teachers should consider before using a manipulative, **how it can enable pupils to eventually do the maths *without* it.** When moving away from manipulatives, pupils may find it helpful to draw diagrams or imagine using the manipulatives.

Effective use of Manipulatives and Representations

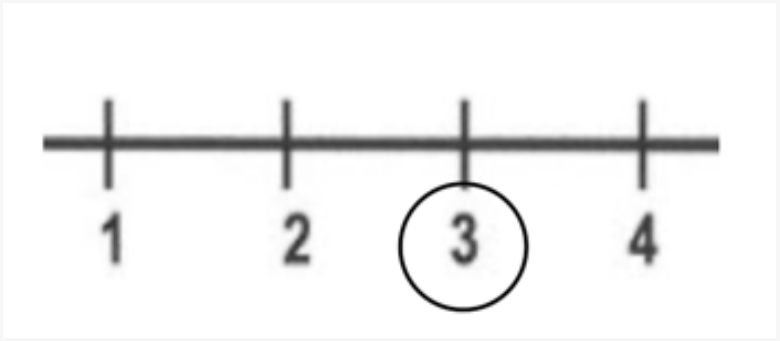
Manipulatives can be used to support pupils of all ages.

The decision to **remove a manipulative** should be made in **response to the pupils' improved knowledge and understanding** and not due to a pupils age.

Making Connections Between Manipulatives and Representations

Evidence shows the importance of children **using different representations of number and making connections between them** in order to support a fuller understanding.

An example could be- Representing aspects of ‘three’.



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Implications of Manipulatives

The use of **multiple representations** has a positive impact on attainment.

Comparison and discussion of different representations helps pupils develop conceptual understanding.

Purposefully select different representations of key mathematical ideas to support pupils development of more abstract, diagrammatic representations.

Be aware that using too many representations at one time may **cause confusion** and hinder learning.

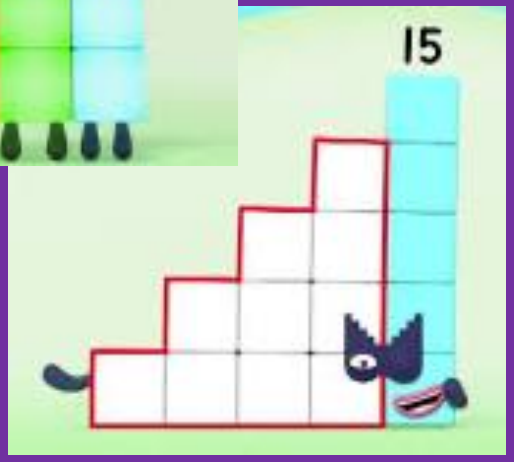
Implications of Manipulatives

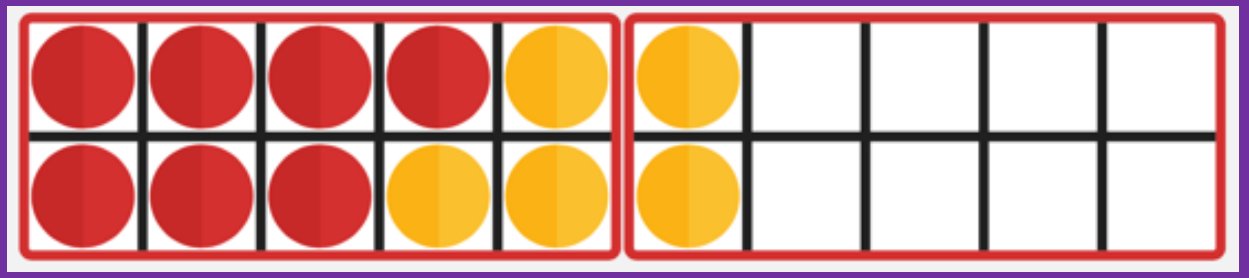
Teachers understanding of mathematical concepts needs to be strong.

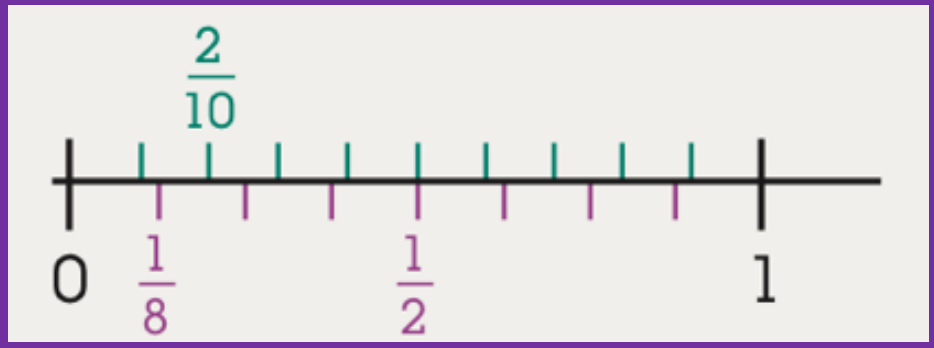
Needs to be a **focus of Continual Professional Development** with manipulatives for all practitioners within school.

Plan the use of manipulatives and representations to ensure a consistent approach throughout the school.

Examples of Good Practice







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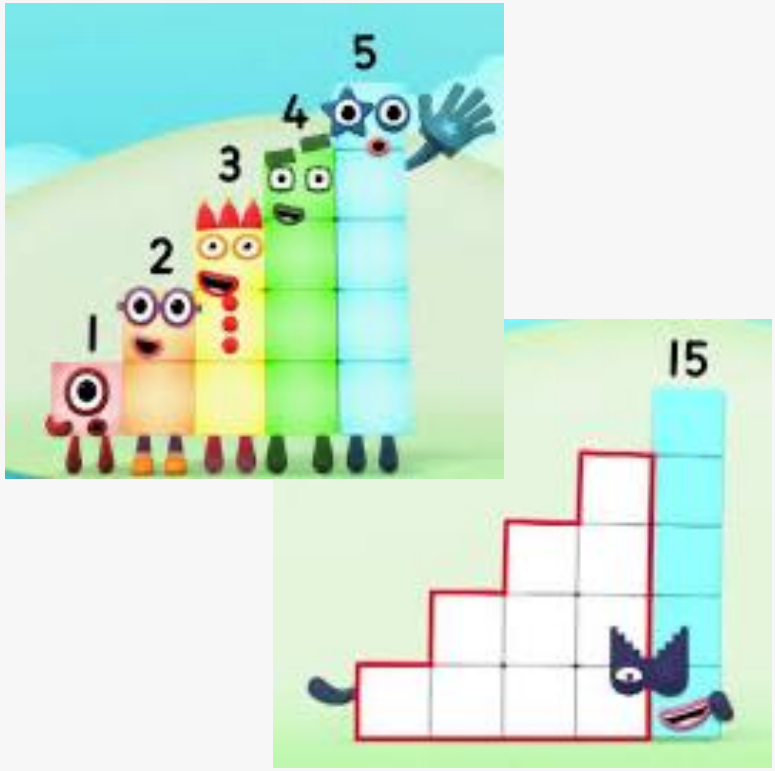
Examples of Good Practice- EYFS

Using manipulatives to explore ‘one more than’

Explore the ‘one more than’ relationship



Children build ‘staircases’, making each ‘stair’ by matching the previous one, then adding one.

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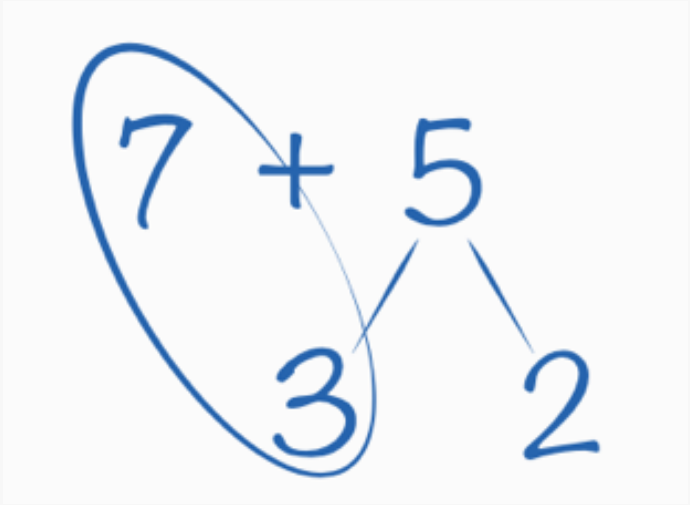
Examples of Good Practice- Year 1

Using manipulatives to explore bridging through ten

7 + 5



$7 + 3 + 2 = 10 + 2$
$= 12$

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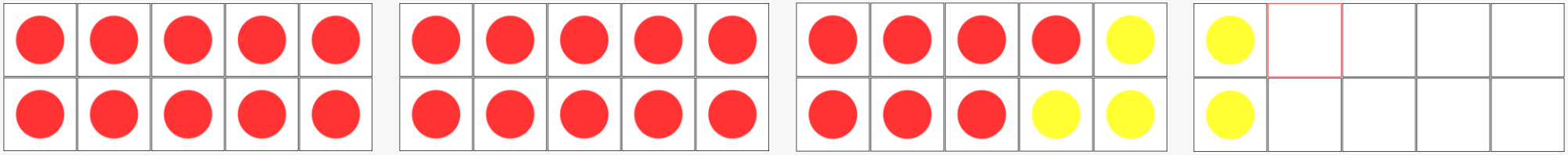
Developing a Secure Conceptual Understanding: the Building Blocks

Using manipulatives to explore bridging through ten in Year 1 is a crucial stepping stone to more complex addition (and subtraction) calculations in later years, for example:

27 + 5



https://mathsbot.com/manipulatives/tenFrame

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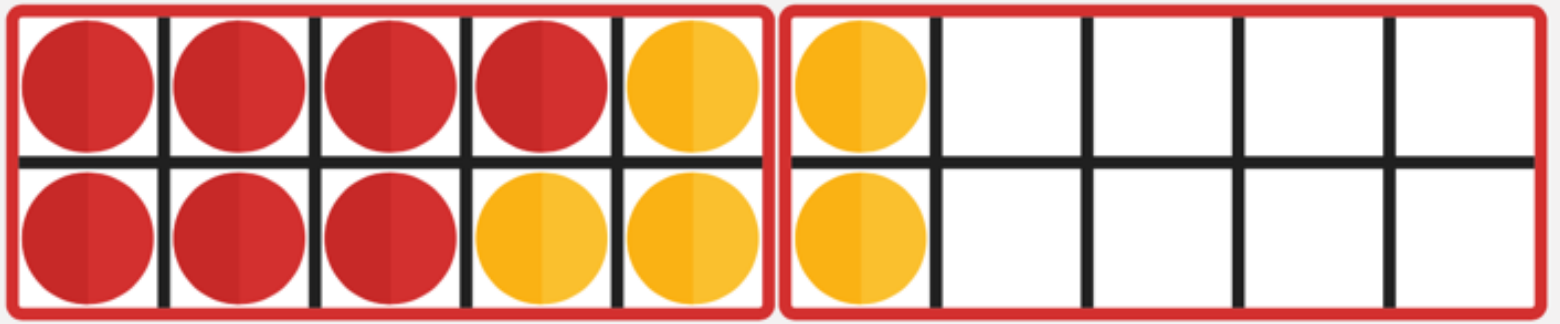
Developing a Secure Conceptual Understanding: the Building Blocks

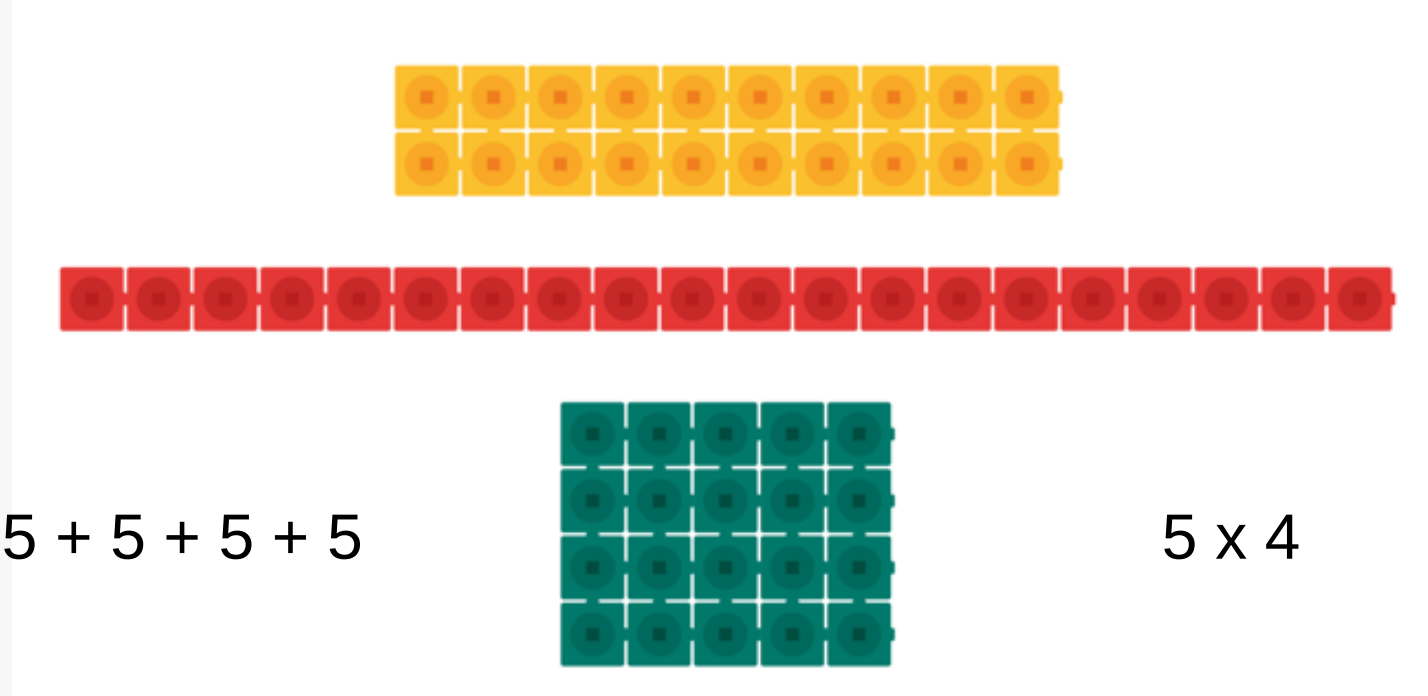
$$^{7}/_{10} + {}^{5}/_{10}$$



Examples of Good Practice- Year 2

Using manipulatives to explore making rectangles

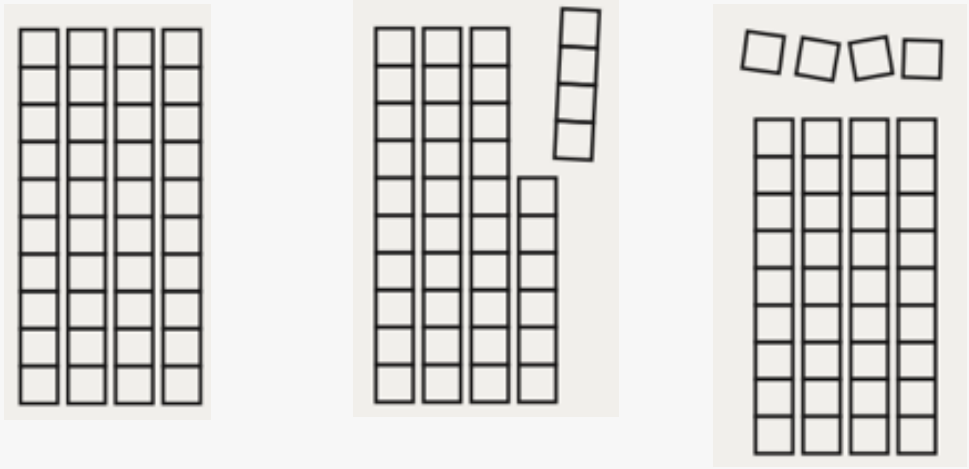
Multiplicative composition of numbers



Examples of Good Practice- KS2

Using manipulatives- supporting finding patterns

40- 4= 36 30-3=27 90-9=81



$40 - 4 = 10\times4 - 1\times4 = 9\times4$

$70 - 7 = 10\times7 - 1\times7 = 9\times7$

$$10t - t = (10 - 1)t = 9t$$

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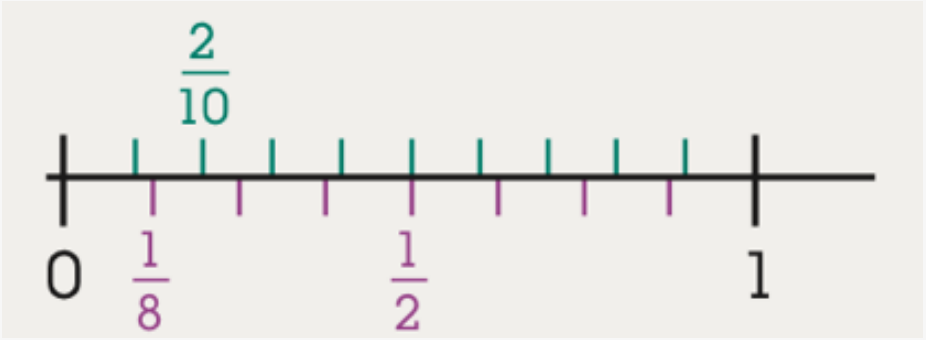
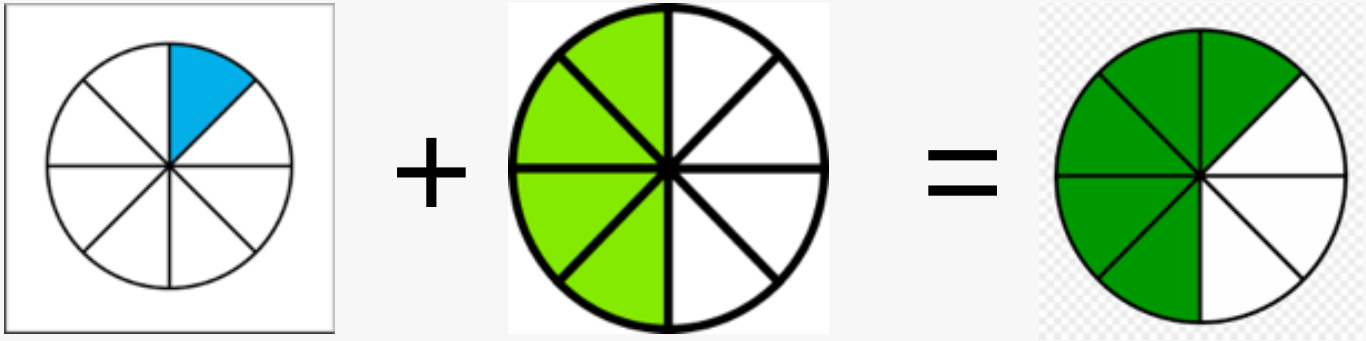
Examples of Good Practice- KS2

Using a number line to support with fraction misconceptions

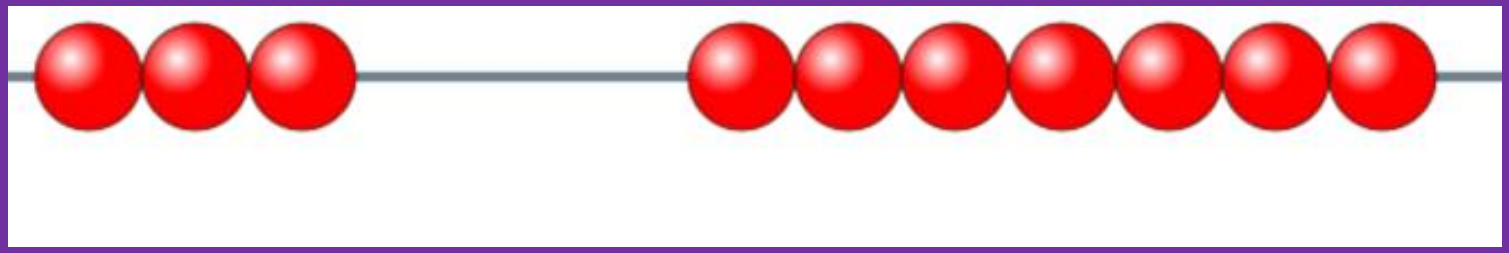
Is this true?

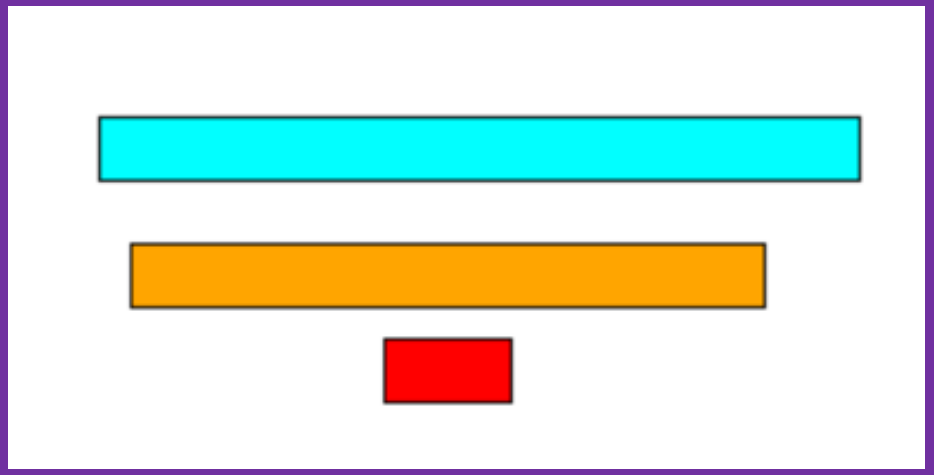
$\frac{1}{2} + \frac{1}{8} = \frac{2}{10}$

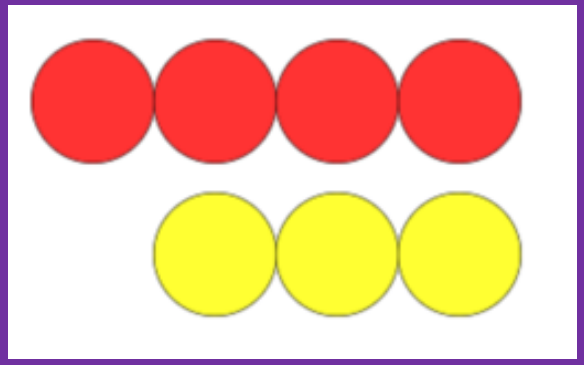




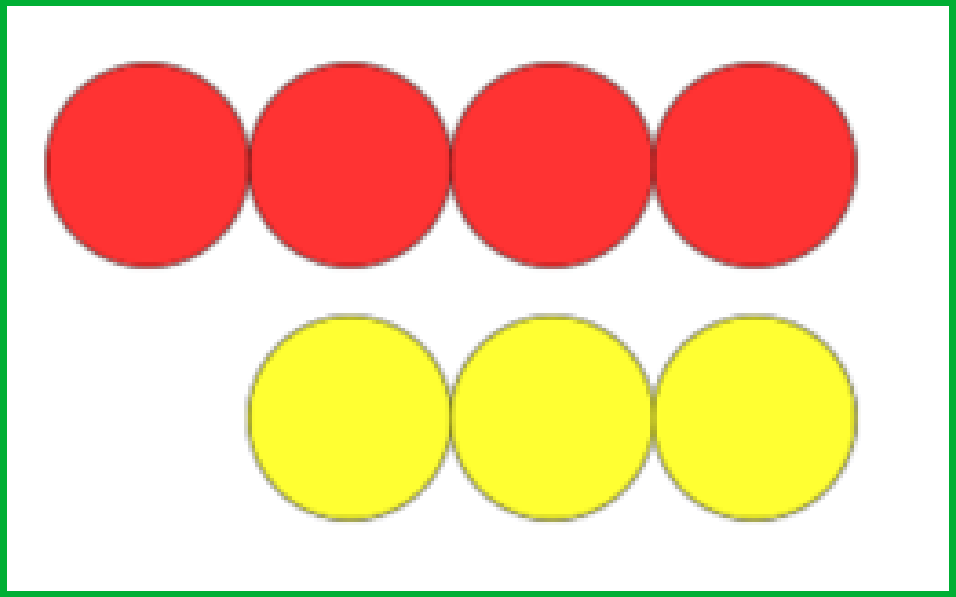
Other Ways to use…







Double Sided Counters



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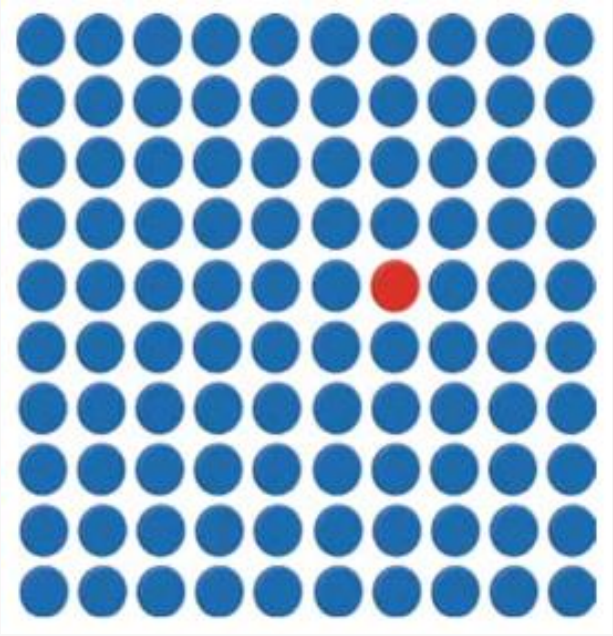
Using Double Sided Counters

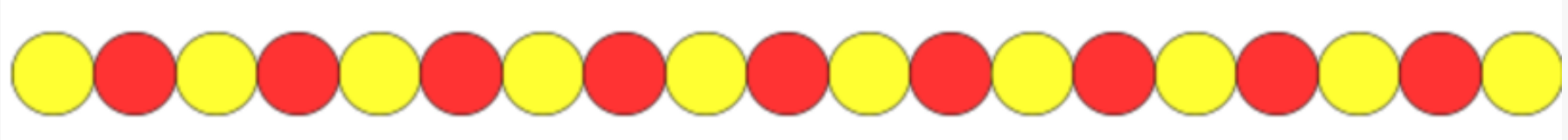
Finding the missing numbers



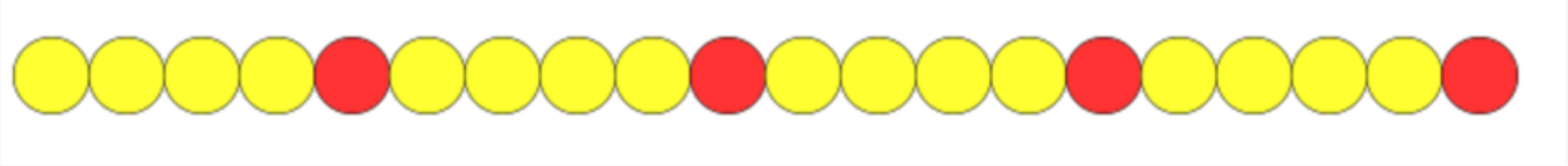
Can be linked to 100 square representation

Using Double Sided Counters

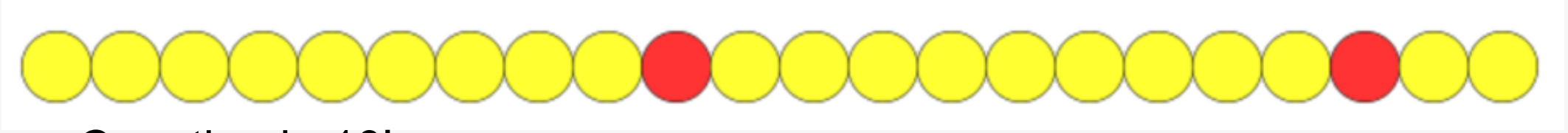
Skip counting of 2, 5 or 10's by turning over the corresponding counter when counting.



Counting in 2's



Counting in 5's



Counting in 10's

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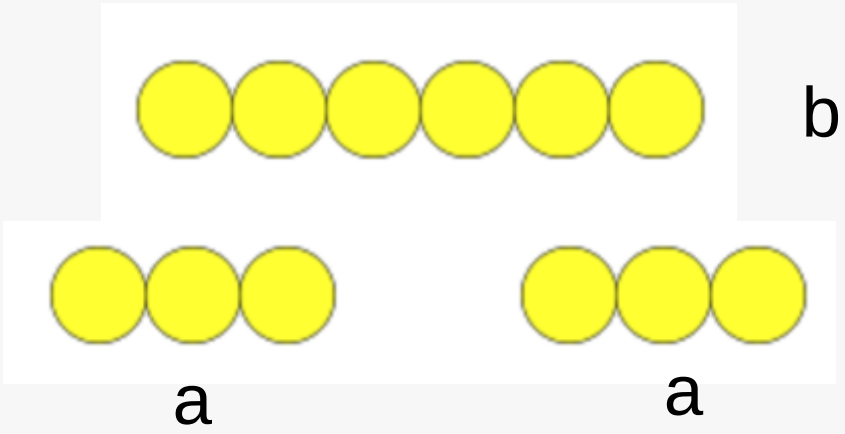
Using Double Sided Counters

Algebra

2a = b

If b = 6, What is a?

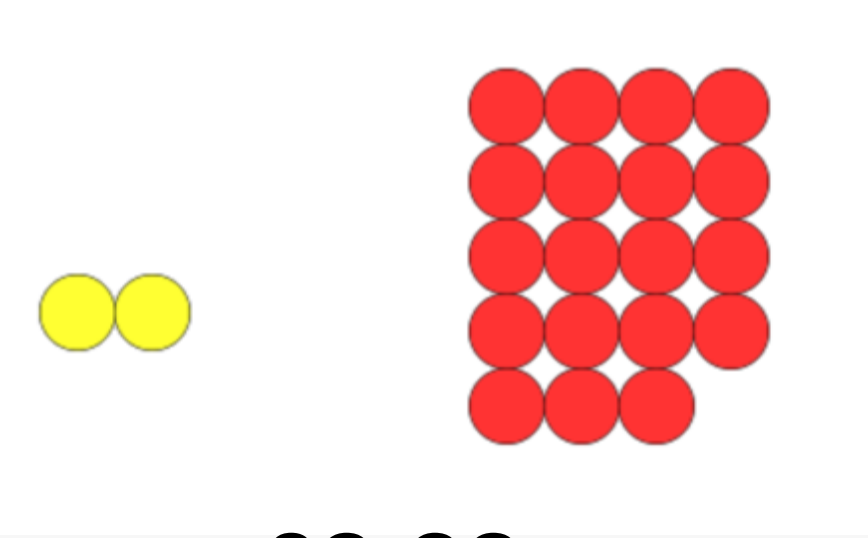
a=3



Using Double Sided Counters

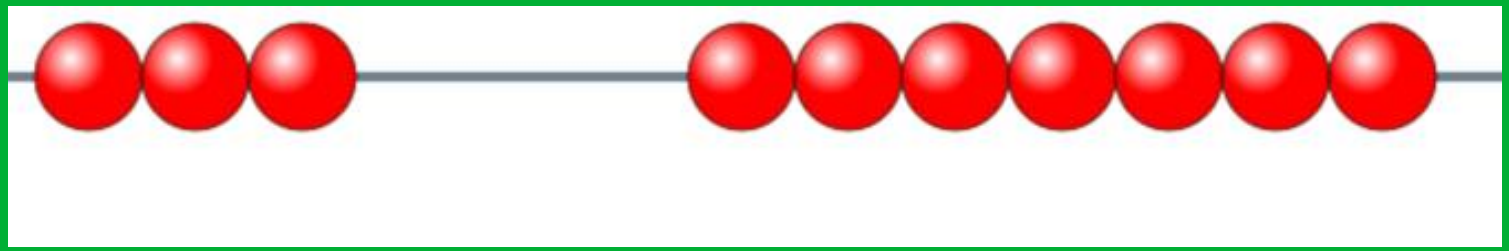
Money

One colour counter to represent pounds and the other colour counters to represent pence. Show an amount of money.



£2.23

Bead Strings



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Using Bead Strings

Showing the relationship between number bonds to 10 and other number bonds.

3+7=10

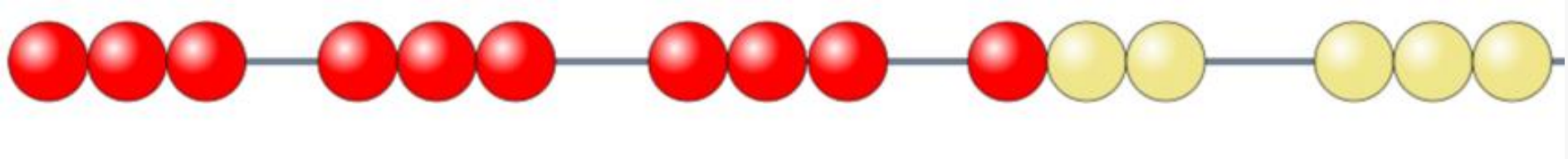
33+7=40

30+70=100

Using Bead Strings

Showing repeated addition.

Great way for children to see when a number goes past a 10 boundary.



3+3+3+3+3=15

10+5=15

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Using Bead Strings

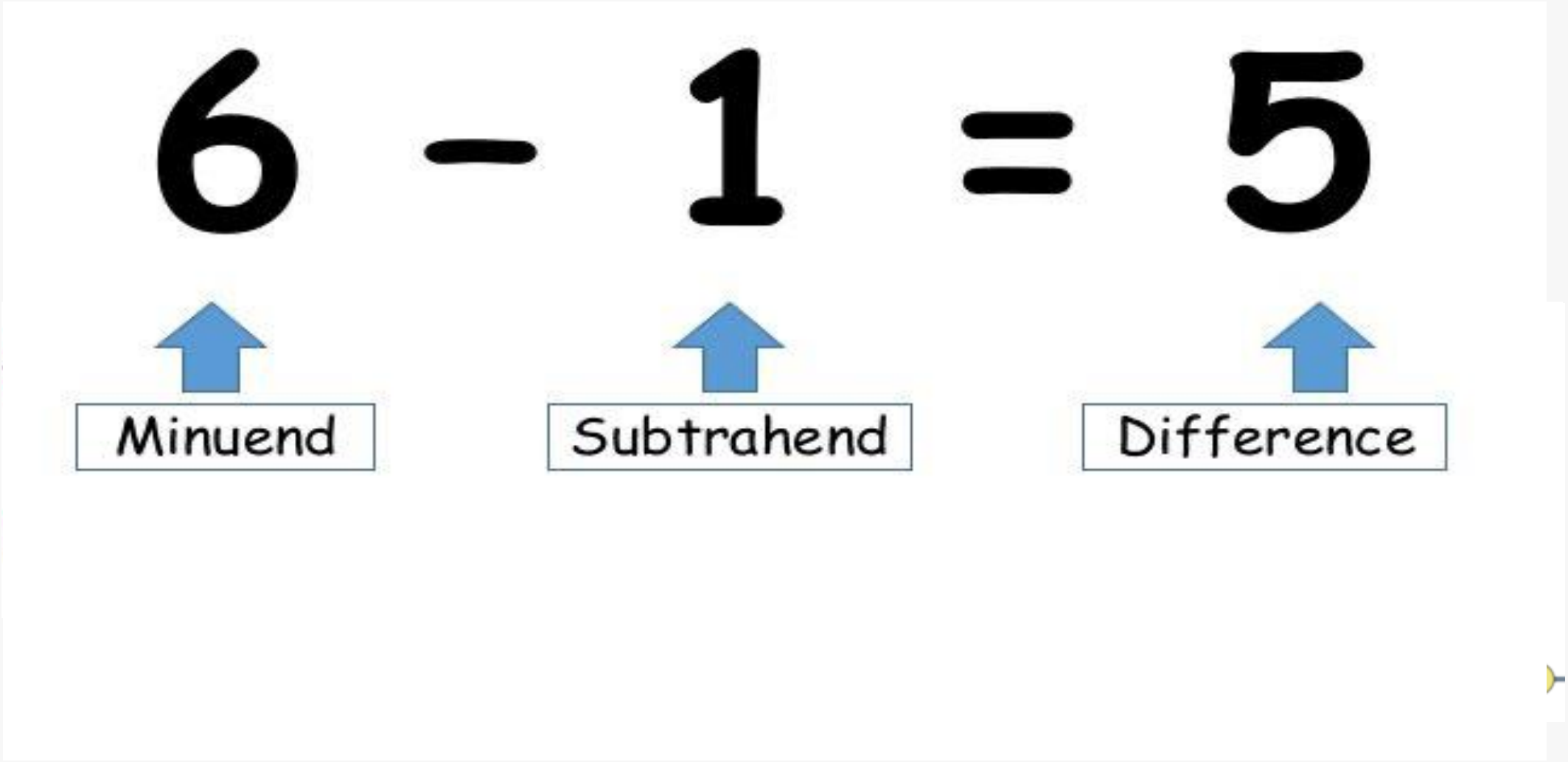
Subtracting a number of beads from a large number.

40

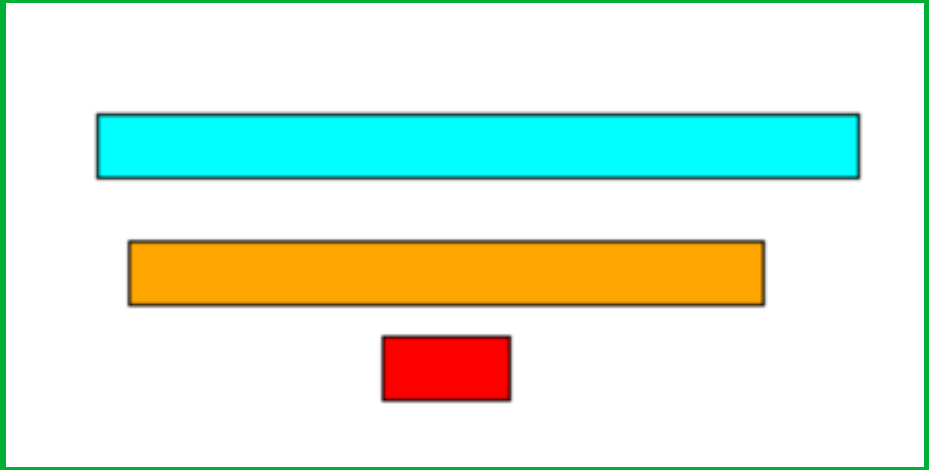
40-5

40-5=35

Using Bead Strings

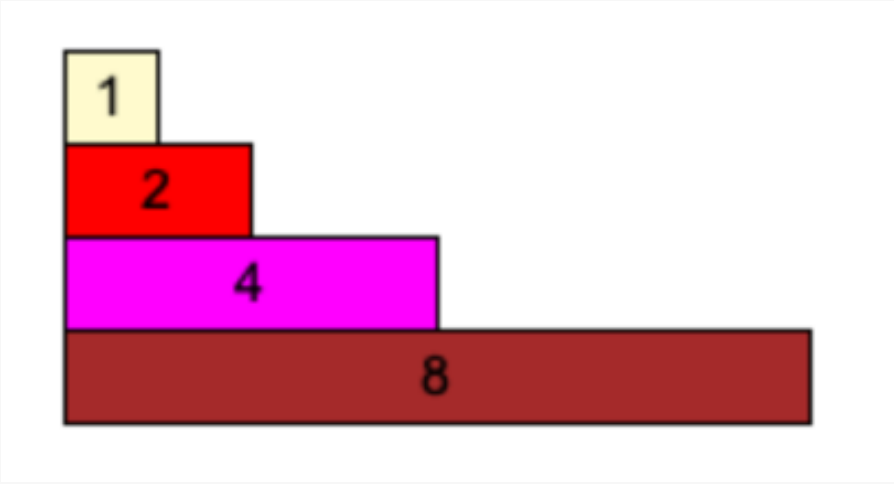


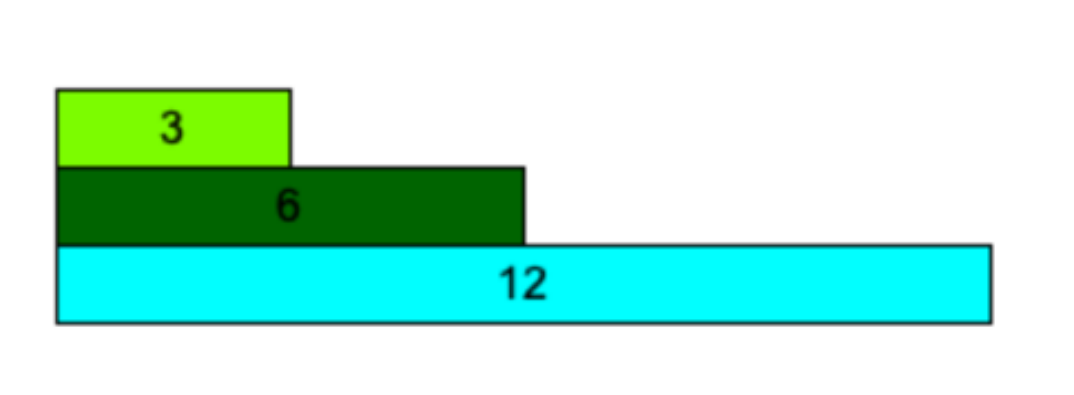
Cuisenaire



Using Cuisenaire

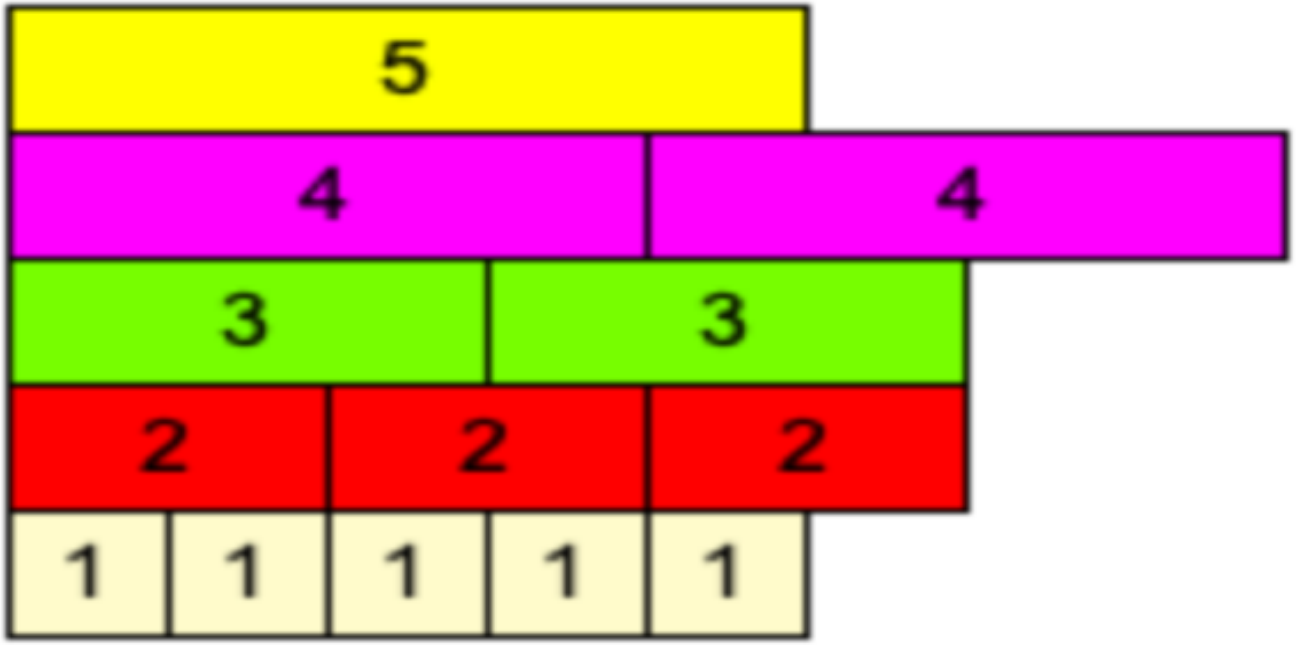
Model the relationship of doubles of numbers.





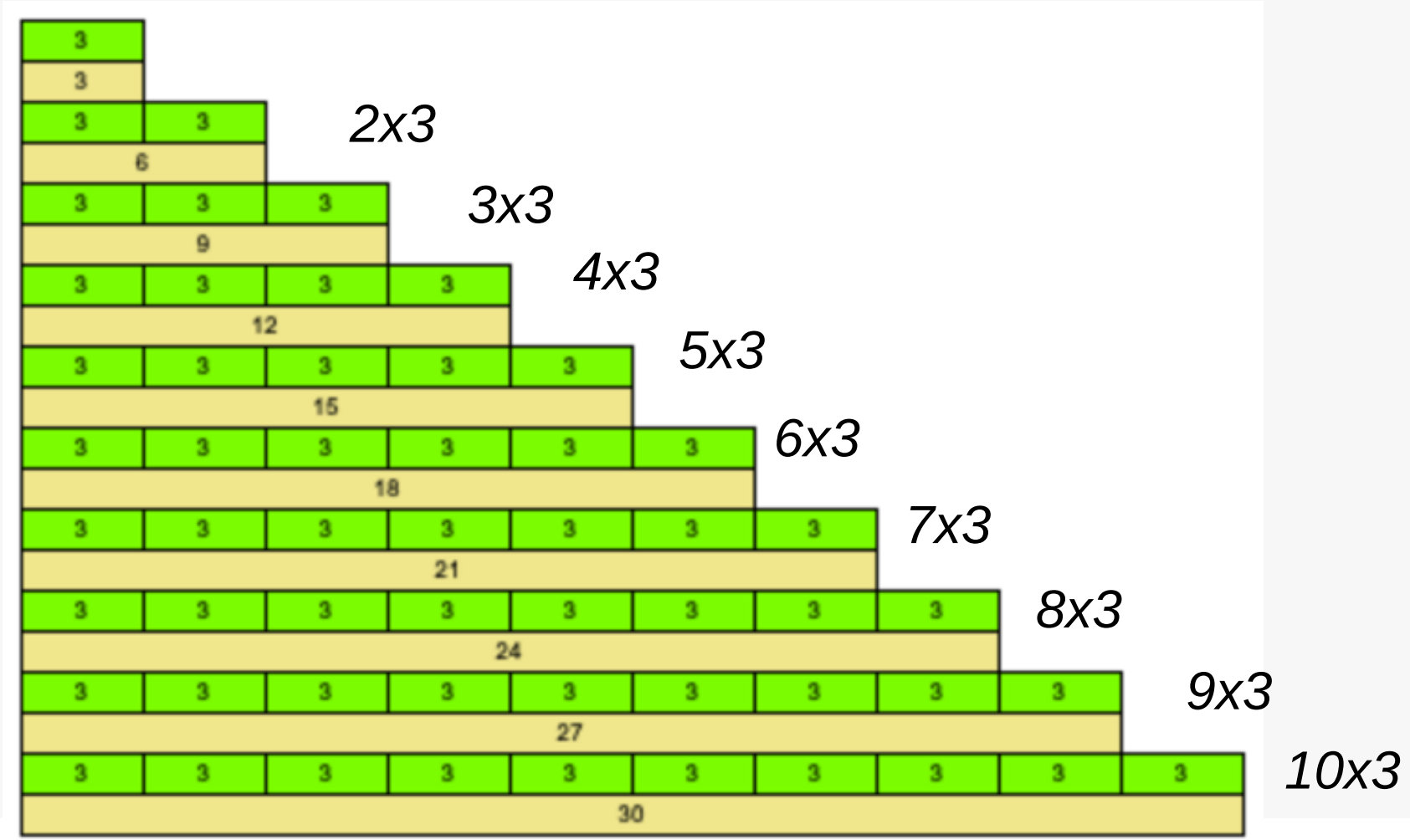
Using Cuisenaire

A good way to explore Prime numbers



Using Cuisenaire

Create a visual representation of a times table e.g. 3



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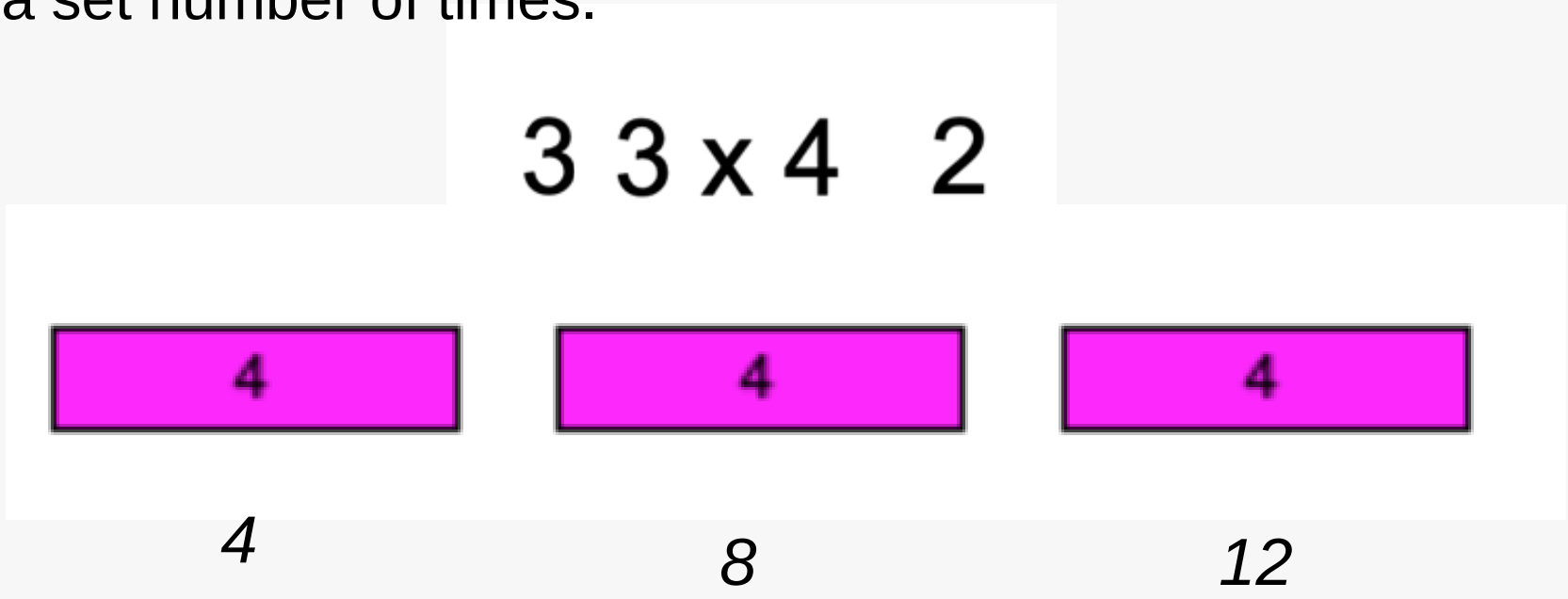
Using Cuisenaire

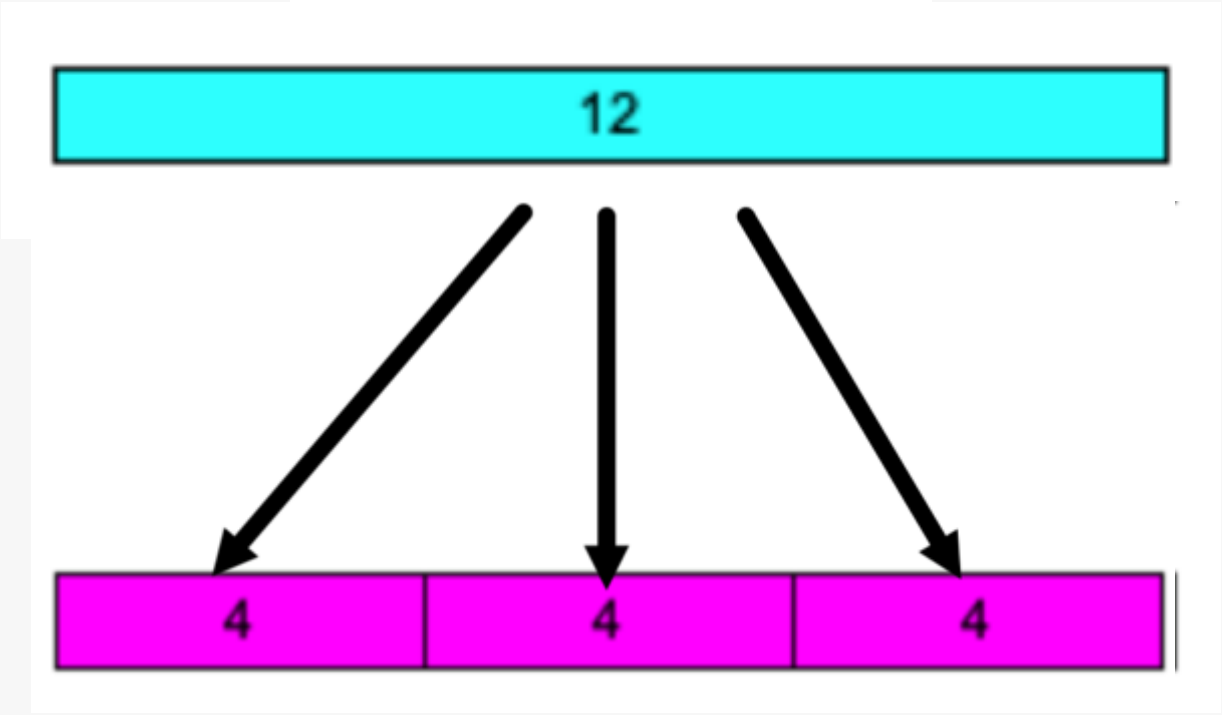
Demonstrate multiplication by repeating the same number of rods a set number of times.



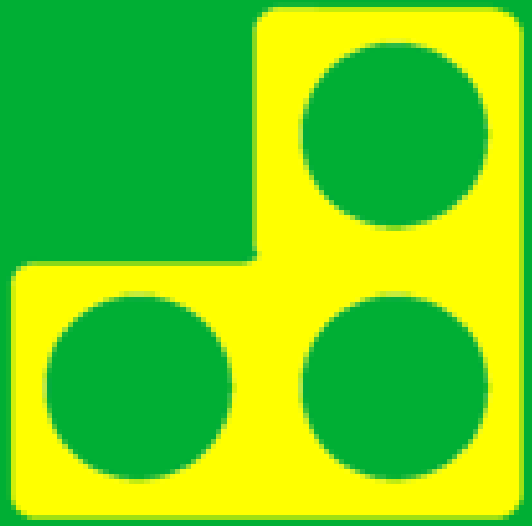
Using Cuisenaire

Show division by making a set number and separating it into groups.

$$12 \div 4 = 3$$



Numicon



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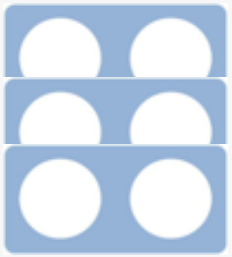
Using Numicon

Ratio: Stack the numicon

~~15~~ 10

6 Girls

9 Boys



2 : 3

11 parts

There are 7 milk chocolates to every 4 dark in a box of 33 chocolates. How many milk and dark chocolates are in the box?

Word Problems (ratio & proportion) Year 6 11

33

dark | 3 | 3 | 3 | 3 | = 12

milk | 3 | 3 | 3 | 3 | 3 | 3 | 3 | = 21

33 ÷ 11 = 3

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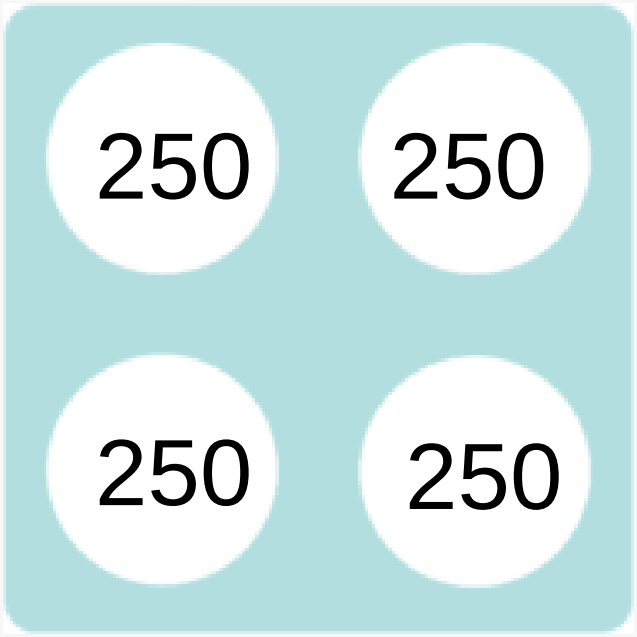
Using Numicon

Multiplying fractions by whole numbers:

$\frac{3}{4}$ of 1000 is 750

250+250+250=750

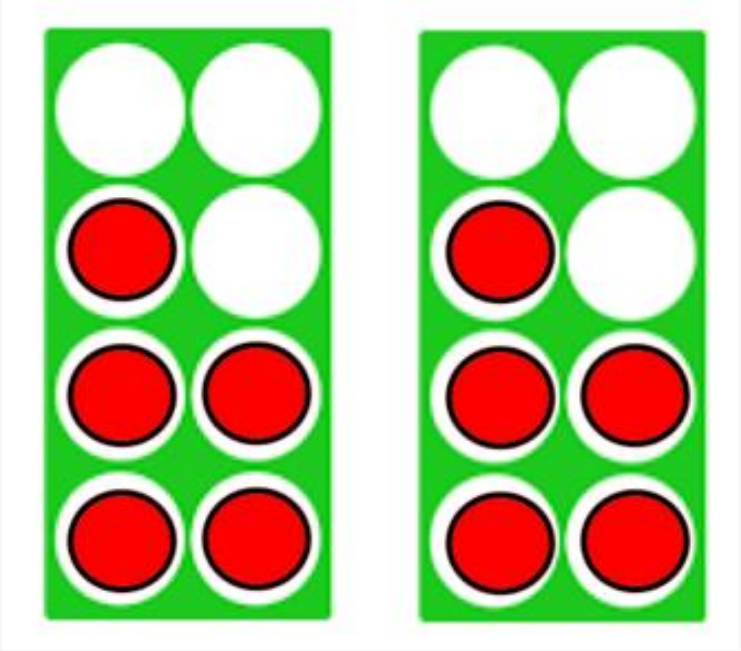


= 1000

Using Numicon

Multiplying fractions by whole numbers:

$\frac{5}{8} \times 2 = 1.25$



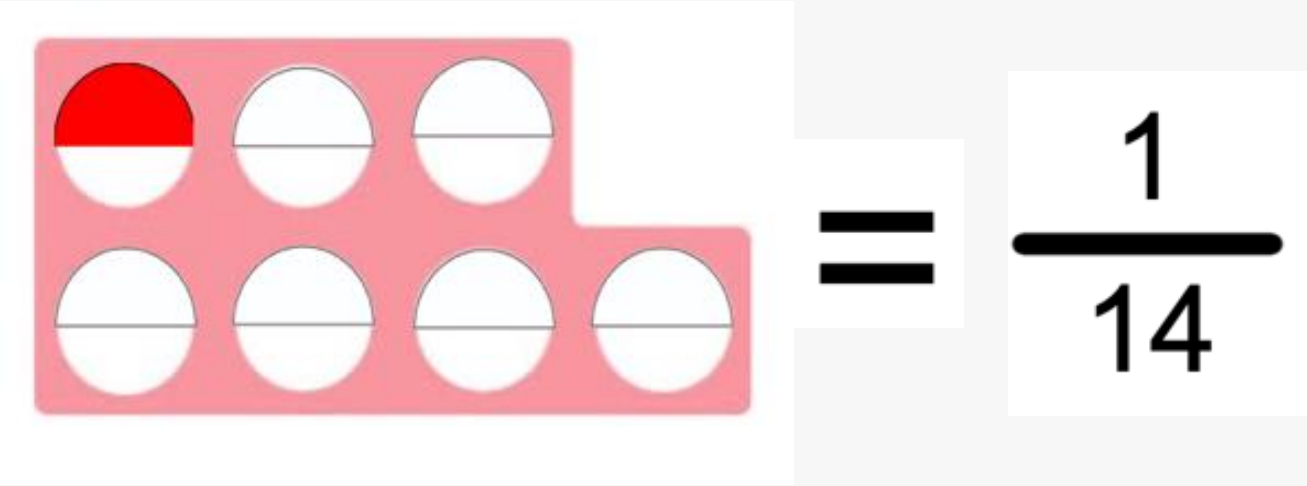
$1\frac{1}{4} = 1.25$

Using Numicon

Multiplying two fractions:

A half multiplied by one seventh is the same as half OF one seventh.

$$\frac{1}{2} \times \frac{1}{7} = \frac{1}{14}$$



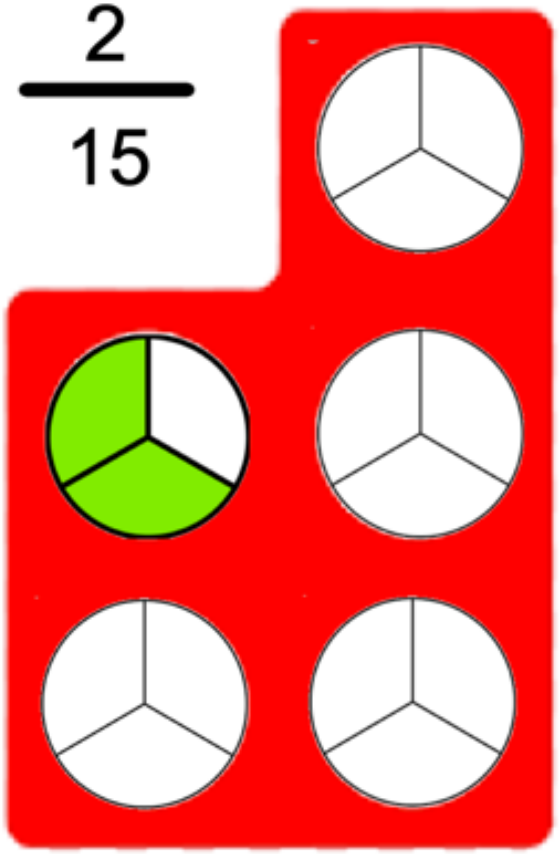
$$= \frac{1}{14}$$

Using Numicon

Multiplying two fractions:

Two thirds multiplied by one fifth is the same as two thirds of one fifth.

$$\frac{2}{3} \times \frac{1}{5} = \frac{2}{15}$$



$$= \frac{2}{15}$$

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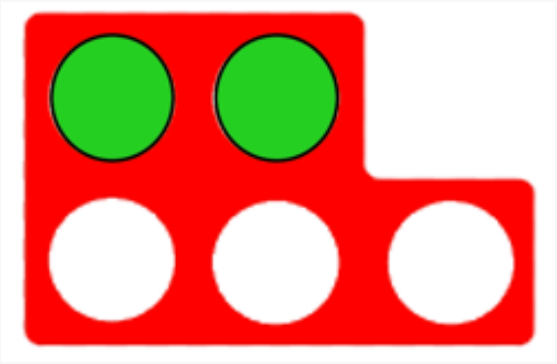
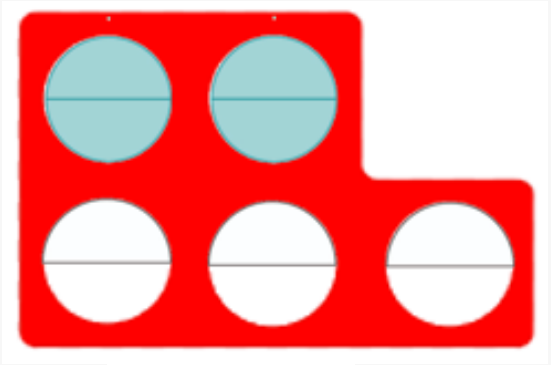
Using Numicon

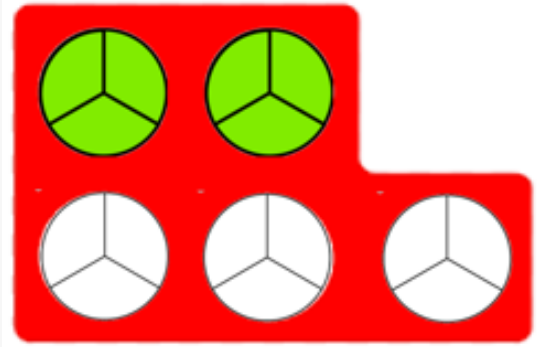
Equivalent fractions:



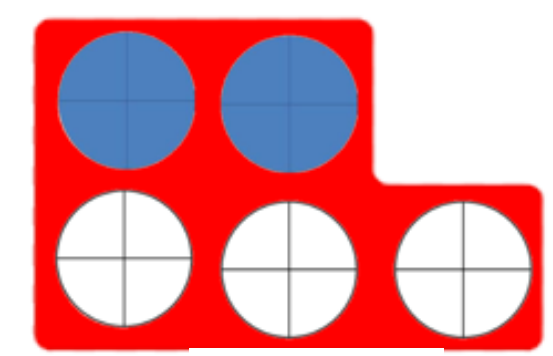
$\frac{2}{5}$



$\frac{4}{10}$



$\frac{6}{15}$



$\frac{8}{20}$

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Concrete Pictorial Abstract

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What is Concrete Pictorial Abstract?

Concrete Pictorial Abstract (CPA) uses physical and visual aids to build a child’s understanding of abstract topics.

Reinforcement of learning is achieved by going **back and forth between representations**, building pupils' **conceptual understanding** instead of an understanding based on **completing mathematical procedures.**

What is Concrete Pictorial Abstract?

Mathematical concepts are introduced through the use of **concrete** resources. This is further explored with pictures – usually **pictorial representations** of the concrete objects. Finally the concept is shown in **abstract form** i.e. numbers or other symbols.

This supports the relationship between numbers and the real world and secures the understanding of the mathematical concept being taught.

Concrete Pictorial Abstract and Jerome Bruner

The origins of the CPA approach originate from Jerome Bruner in the 1960s as a means of **scaffolding learning**.

The abstract nature of learning needs to be **scaffolded** by the use of **effective representations and maths manipulatives**.

The CPA approach allows pupils to build upon mathematical understanding of the concepts being learned, which in turn leads to **information and knowledge being internalised**.

What is a ‘Concrete’ Representation in CPA?

New concepts are introduced using **physical objects or practical equipment.**

Physically handling objects enabling children to explore different **mathematical concepts through exploration**. These can be manipulatives as well as ordinary household items.

The use of concrete materials allows children to **see’ and make sense of what happening in a concept.**

What is a ‘Concrete’ Representation in CPA?

All children, **regardless of ability, benefit from practical resources**, ensuring understanding goes beyond the learning of a procedure.

Practical resources develop reasoning and discussion, as children can **articulate and explain a concept**. Teachers can understand where misconceptions lie and to establish the depth of pupils understanding.

What is a ‘Pictorial’ Representation in CPA?

Children progress to drawing pictorial representations or quick sketches of the objects alongside the manipulatives. This enables pupils to link the concept as well as benefit from the visual support the resources provide.

Pictorial recording is key to ensuring that children can make the **link between a concrete resource and abstract notation.**

The Pictorial acts as a **bridge** between the **concrete resources** that were being used and the **abstract symbols** that they need to understand.

What is an ‘Abstract’ Representation in CPA?

When a secure understanding of the concept is gained through concrete resources and visual images learners can explore the abstract stage.

Abstract symbols are used to model problems – usually numerals.

To access this stage effectively, children **need access to the previous two stages alongside it**.

For most effective learning, children need to constantly go back and forth between each of the stages to ensures concepts are reinforced and understood.

Effective use of the Concrete Pictorial Abstract Method

A **misconception** with this CPA model is to teach the concrete, then the pictorial and finally the abstract.

All stages should be **taught simultaneously** whenever a new concept is introduced and when exploring the concept further.

When concrete resources, pictorial representations and abstract recordings are all used pupils make strong links between each stage.

Use of CPA Approach

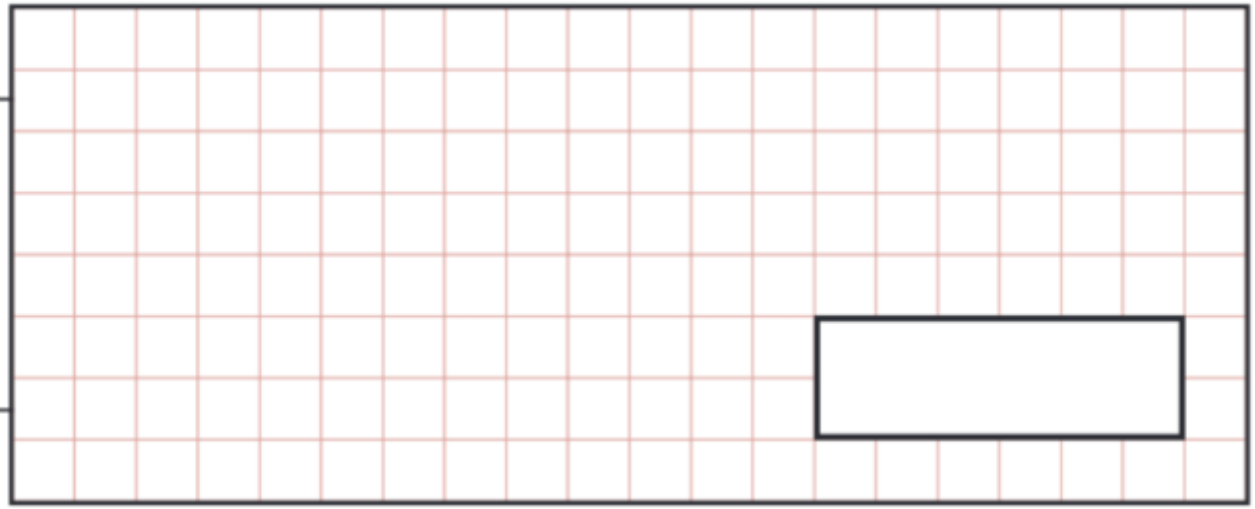
8 At the start of June, there were 1,793 toy cars in the shop.

During June,

- 8,728 more toy cars were delivered
- 9,473 toy cars were sold.

How many toy cars were left in the shop at the end of June?

Show your method



2 marks

Use of CPA Approach

At the start of June, there were 1,793 toy cars in the shop.

During June,

- 8,728 more toy cars were delivered

1,793	8,728

10,521

1000's	100's
1000	100 100 100 100 100 100 100

10000's	1000's	100's	10's	1's
10000		100 100 100 100 100	10 10	1

1's
1 1 1 1 1 1 1 1

Use of CPA Approach

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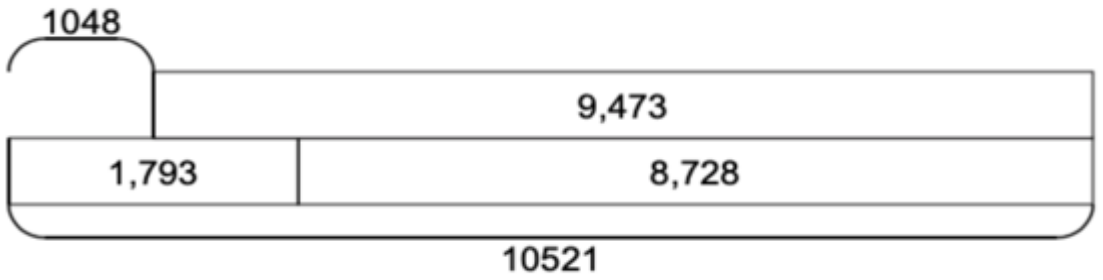
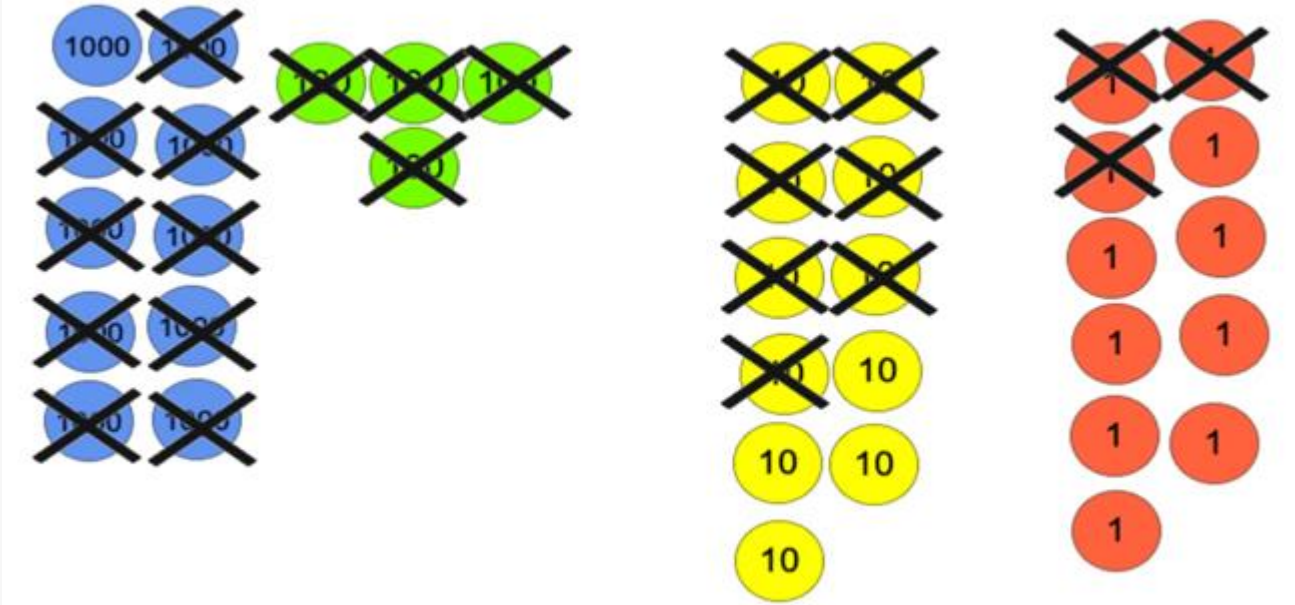
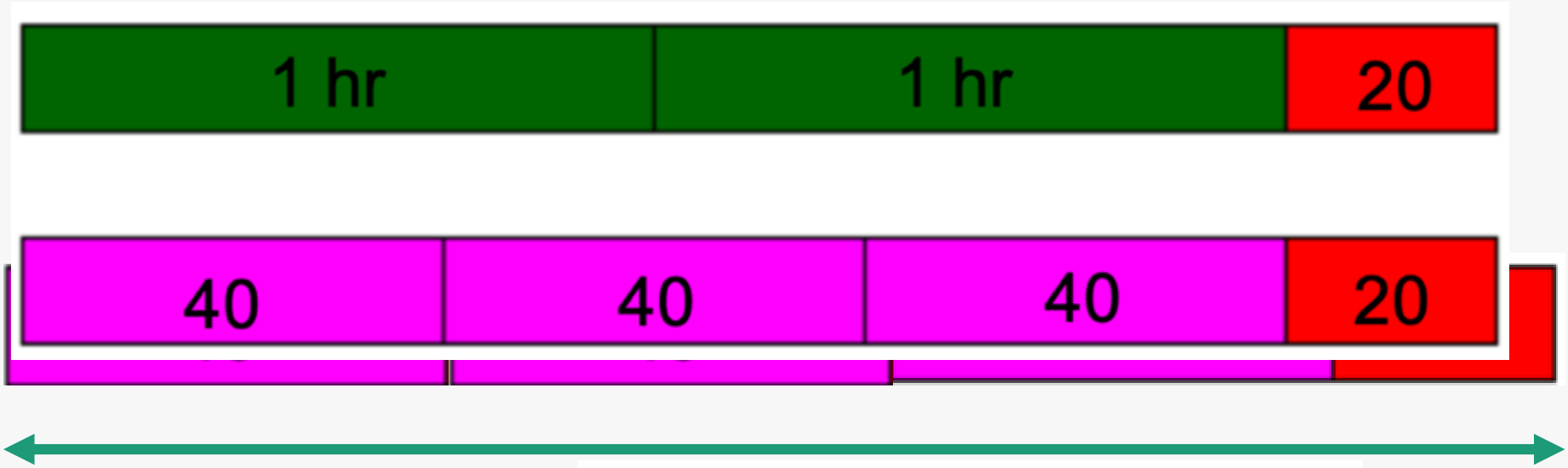




1793+8728-9473=1048

Cooking time = 20 minutes plus an extra
40 minutes for each kilogram

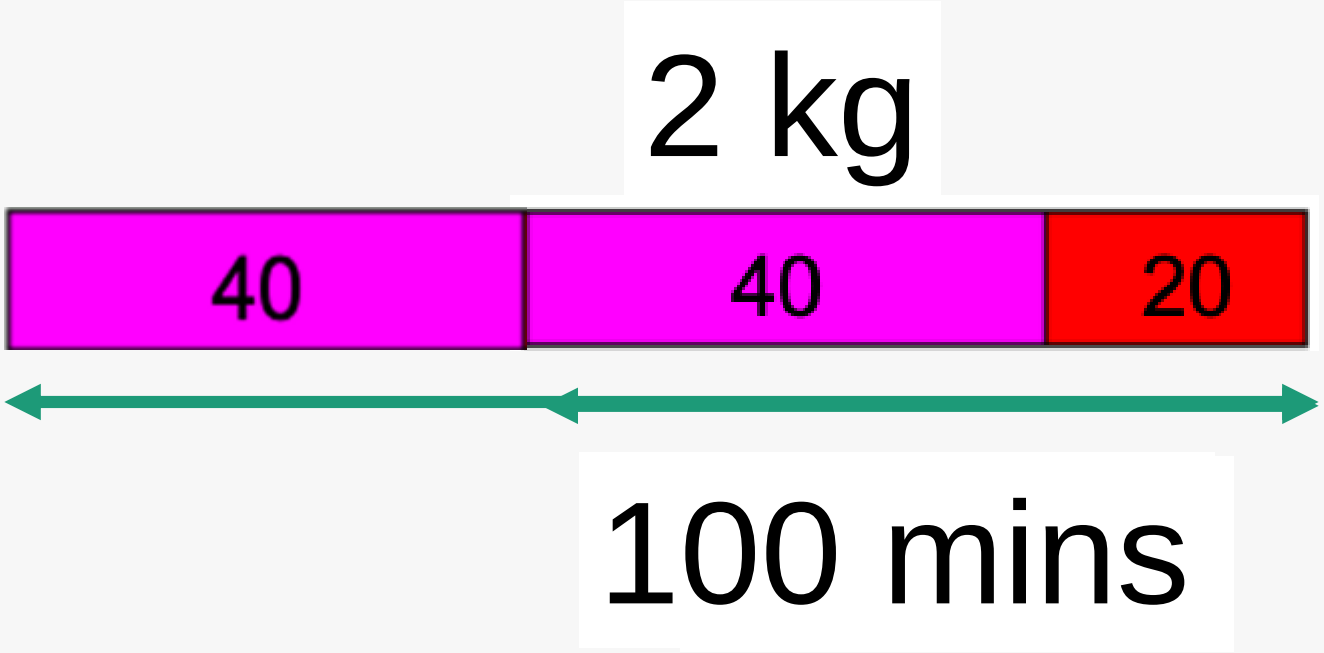
How many minutes will it take to cook a 3 kg chicken?



2 hrs 20 mins

Cooking time = 20 minutes plus an extra
40 minutes for each kilogram

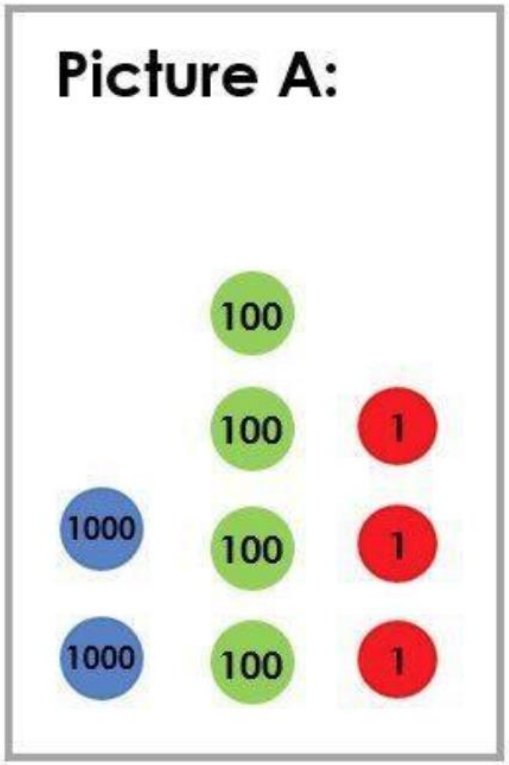
"What is the mass of a chicken that takes 100 minutes to cook?"



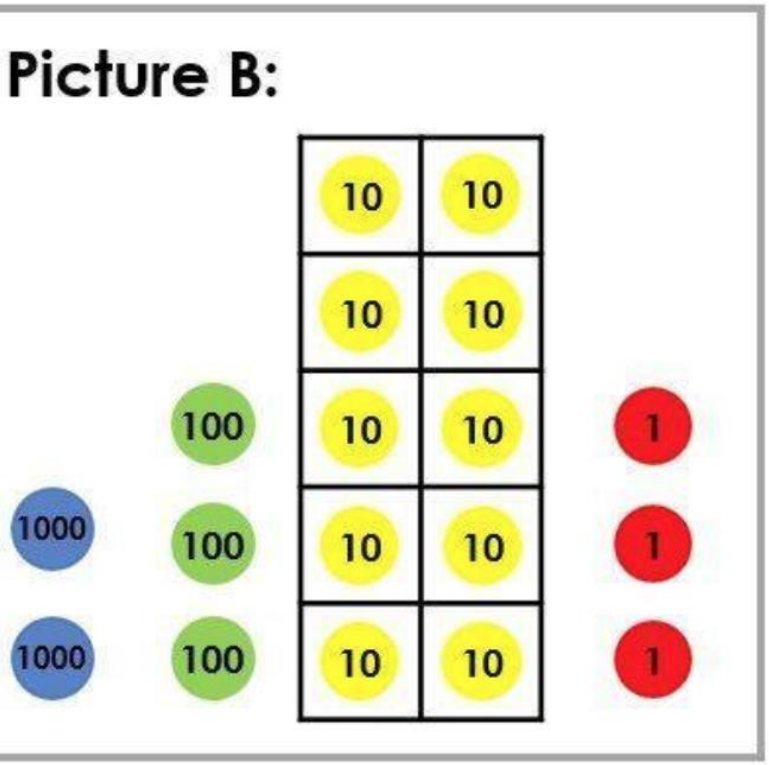
How Is As Important As What!

Part 1: Explain how **both** pictures show **2403**

Picture A:



Picture B:

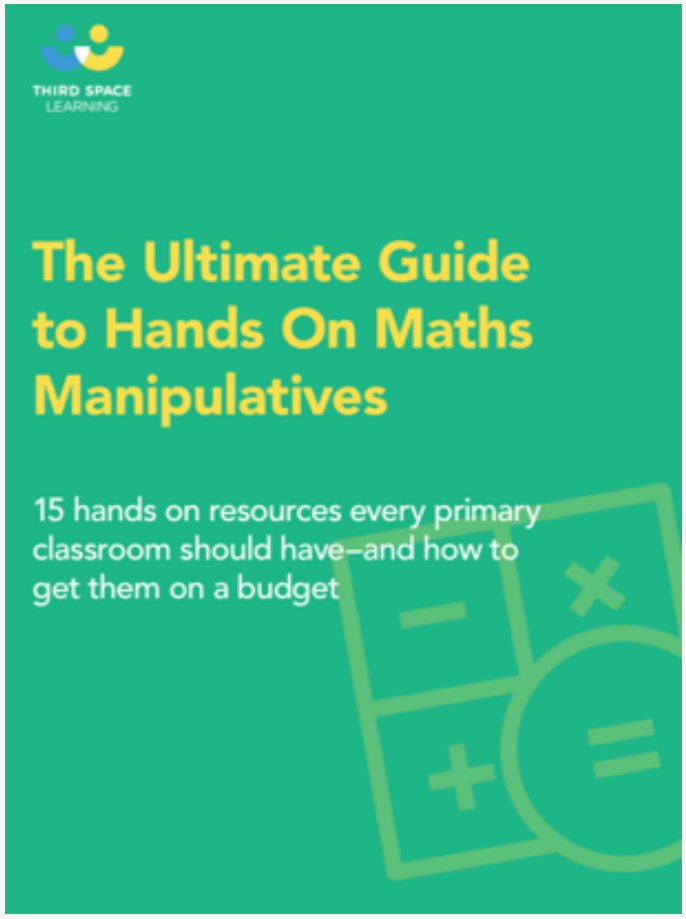


Part 2: Which picture is **most helpful** for calculating **2403 – 10?**

Explain why.

Support- Manipulatives

https://thirdspacelearning.com/resources/resource-ultimate-guide-maths-manipulatives/



Guide to dienes, numicon, clothes pegs, geoboards, arrow cards, straws, dominoes, playing cards, dice, bead strings, Cuisenaire, tens frames, double sided counters, square counters, place value counters.

- Summary
- Suitable for
- Pros
- Cons
- Price
- Where to buy
- Budget alternatives.

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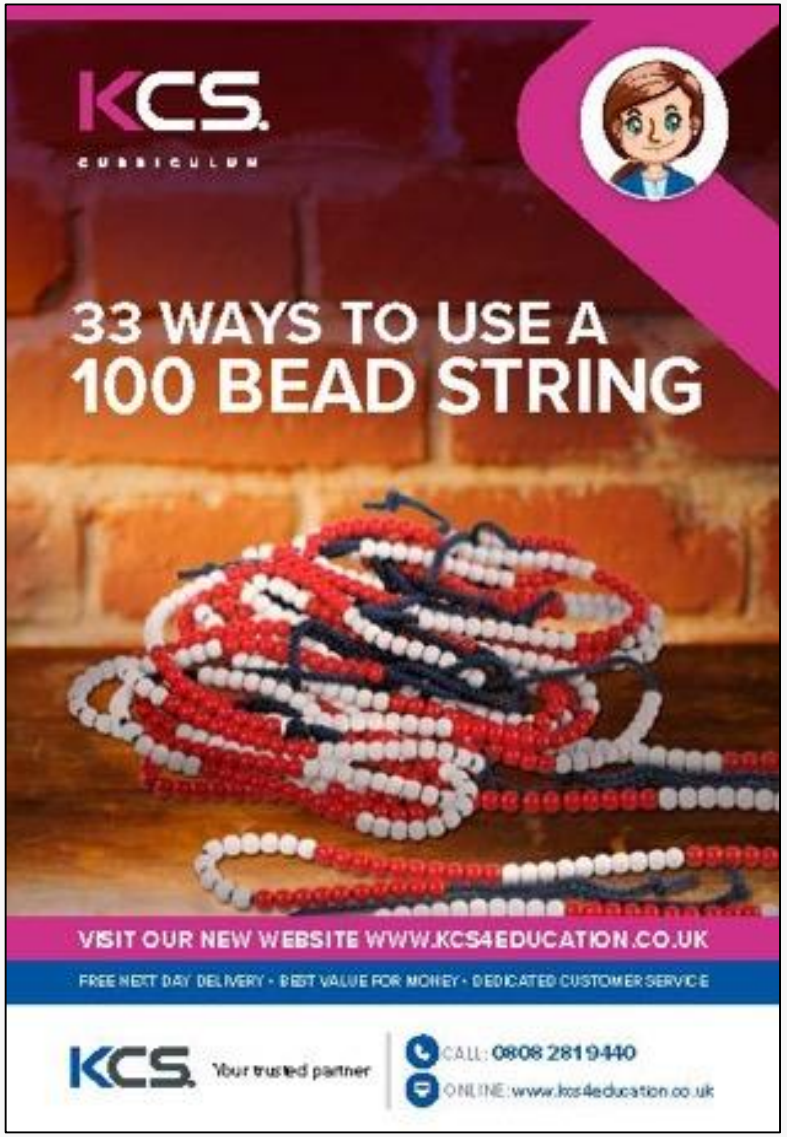
Support - Manipulatives



Ideas for:

- Bead Strings
- Base 10
- Tens Frame
- Place Value Counters
- Number Line
- 100 Square
- Double Sided Counters
- Cuisenaire
- Dominoes.

https://www.kcs.co.uk/inspiration-booklets

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Language Focus

STEM Sentences

Mathematical reasoning and conceptual understanding is enhanced when children use correct mathematical terminology (e.g. saying ‘digit’ rather than ‘number’).

Children should explain their mathematical thinking in complete sentences.

I say, you say, we all say

Sentence stem are provided so children can communicate their ideas with mathematical precision and clarity. Sentence frameworks embed conceptual knowledge and build understanding. Alongside manipulatives and representations the Stem Sentence highlights key conceptual ideas or generalities in the maths.

Language Focus- Dfe Primary National Curriculum Guidance

Language focus

“If the numerators are both 1, then the larger the denominator, the smaller the fraction.”

Occur in every year group chapter and capture key features of the maths, the things that are crucial for pupils to understand and remember.

Develop the ideas and capture them in a sentence.

Repeating them together and individually will draw attention to their importance and help embed in the long-term memory.

Guidance for Working With a Small Pre-teaching or Intervention Group

Engage in the activities, using manipulatives to enhance the interaction and stimulate discussion.

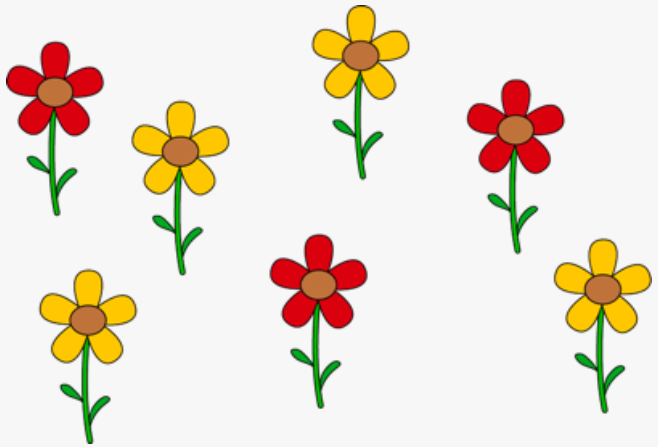
The ultimate aim is to develop fluency in the mathematical ideas such that resources are no longer needed.

Repeat questions, using other numbers/examples where relevant.

Repeat the language structures wherever relevant to build fluency with the key idea and connect the learning. For example:

10 hundreds are equivalent to 1,000.

Read, Write and Interpret Additive Equations



4 + 3

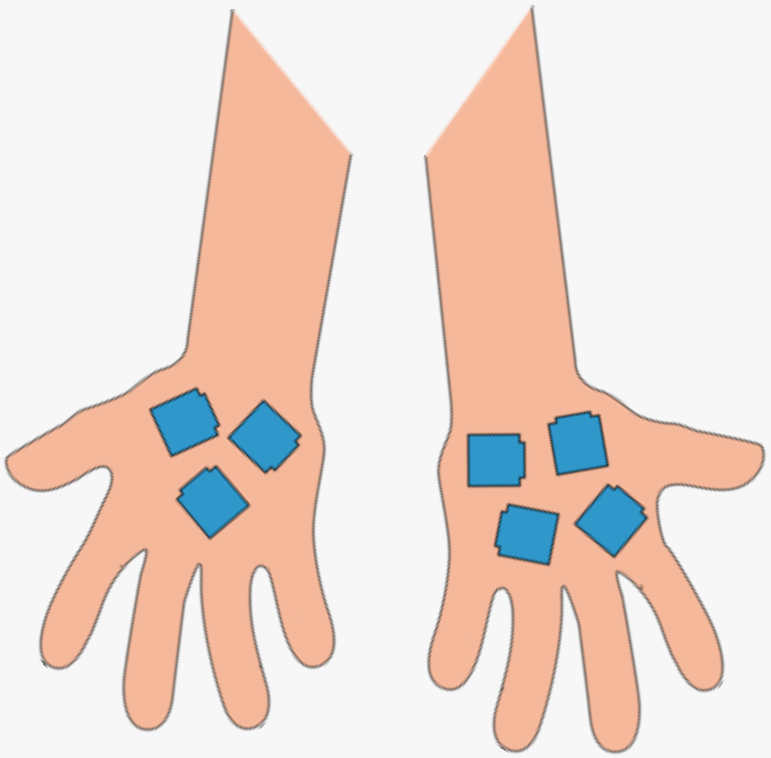
The 4 represents the 4 yellow flowers.

The 3 represents the 3 red flowers.

We can write this as 4 plus 3.

- What do you notice about the flowers?
- How can we write an *expression* using numbers and the addition sign to tell us about the picture?
- Make the addition sign with your fingers and remember to say *plus* or *add* when you see it.
- What does the 4 mean? What does the 3 mean?

Read, Write and Interpret Additive Equations



3 + 4

We can write the addends in either order.

- How can we write an expression to show the cubes in these hands?
- How could we write the expression if the hands switch around?
- Did you see that the position of the *addends* changed? Why did they change?
- Put some cubes in *your* hands. You friend can write an expression, then switch your hands for them to write a new expression!

Read, Write and Interpret Additive Equations

$5 + 2$

$3 + 4$

$2 + 5$

- Look at the rucksacks. How could you sort them in different ways?
- Can you write some expressions using the addition sign to show the different sizes *or* different colours?
- Can you say what the numbers mean in each of my expressions?
- Can you think of another expression we could write?

Read, Write and Interpret Additive Equations

5 + 2 = 7

We can write 5 plus 2 is equal to 7.

The 5 represents the number of flowers in 1 bunch.

The 2 represents the number in the other bunch.

The 7 represents the total number of flowers.

- How many flowers are in each bunch? How many flowers will there be if we *combine* the bunches?
- What *equation* could we write to show this?
- What symbols will we need in our equation? Can you make the *equals* sign with your fingers and remember to say *is equal to* when you see it?
- Can you describe what each number means?

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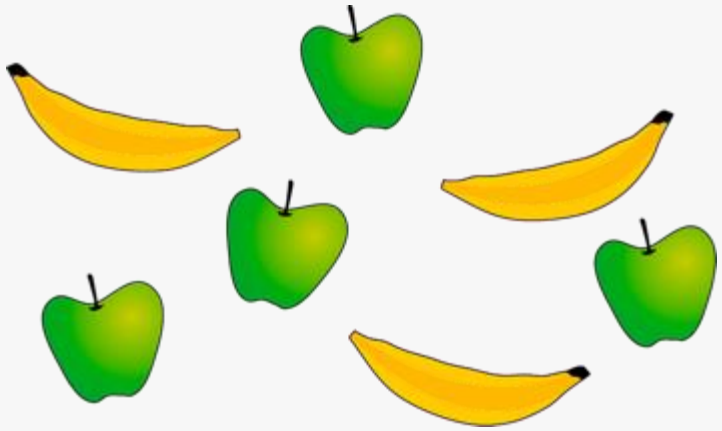
Read, Write and Interpret Additive Equations



$3 + 3 = 6$

$4 + 3 = 7$

$5 = 4 + 1$

$8 = 4 + 3$

- Can you help me decide which of my equations matches the picture?
- Why are some of the equations *not* correct?
- Can you describe what the numbers mean in the correct equation?

The 4 represents the 4 apples.

The 3 represents the 3 bananas.

The 7 represents the total number of fruit.

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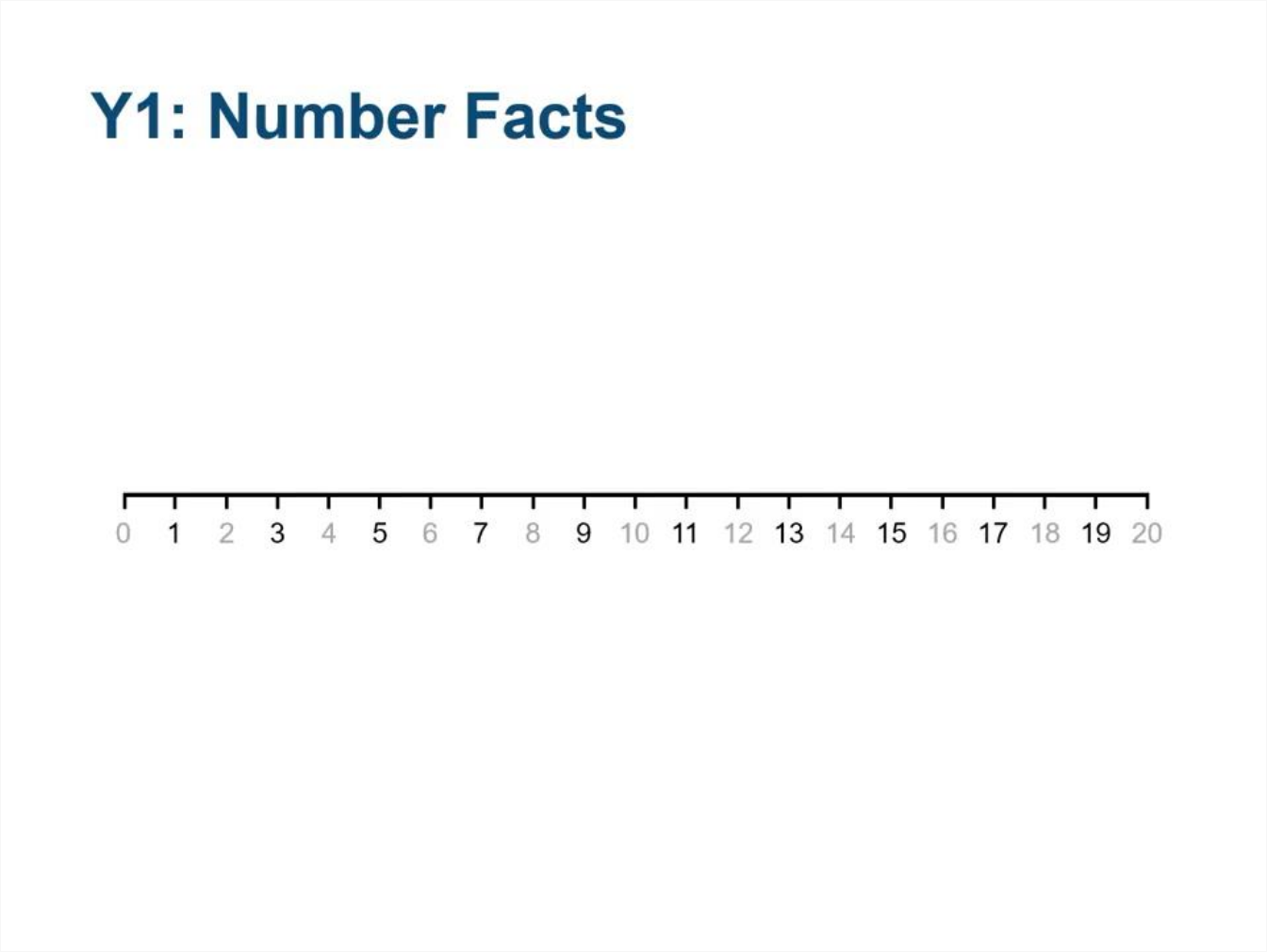
DfE Primary National Curriculum Guidance



https://www.youtube.com/playlist?list=PL6gGtLyXoeq-FMWk00AlcIPo3fhGmi03D

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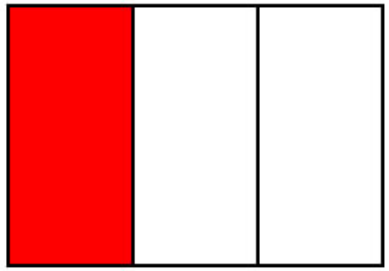
STEM Sentence Examples

If the rectangle is the whole, the shaded part is one third of the whole.



There are 12 **stars**. $\frac{1}{3}$ of the **stars** is equal to **4 stars**

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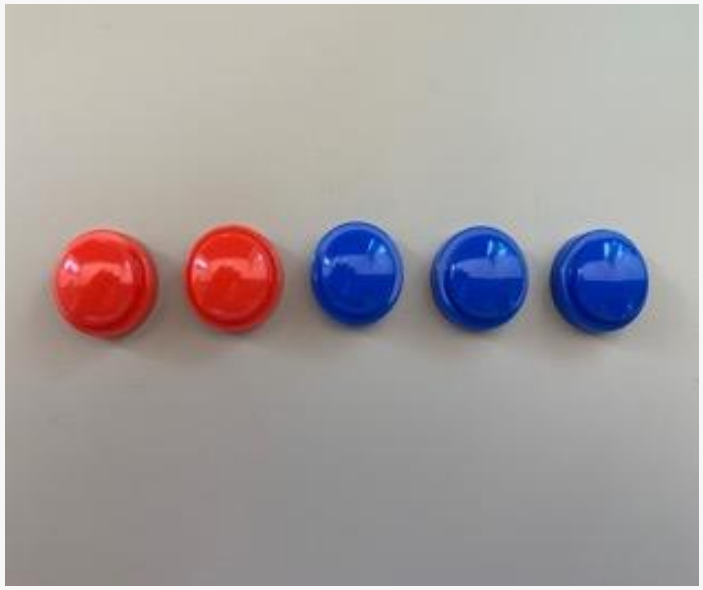
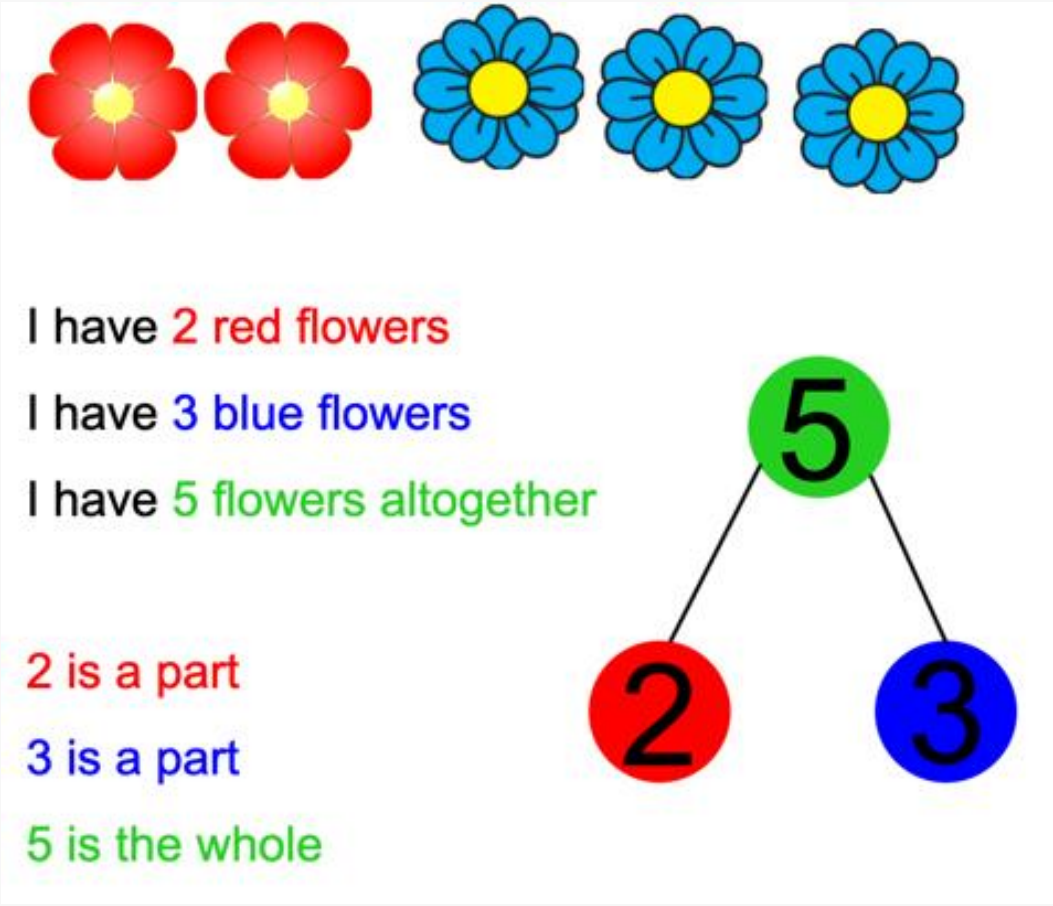
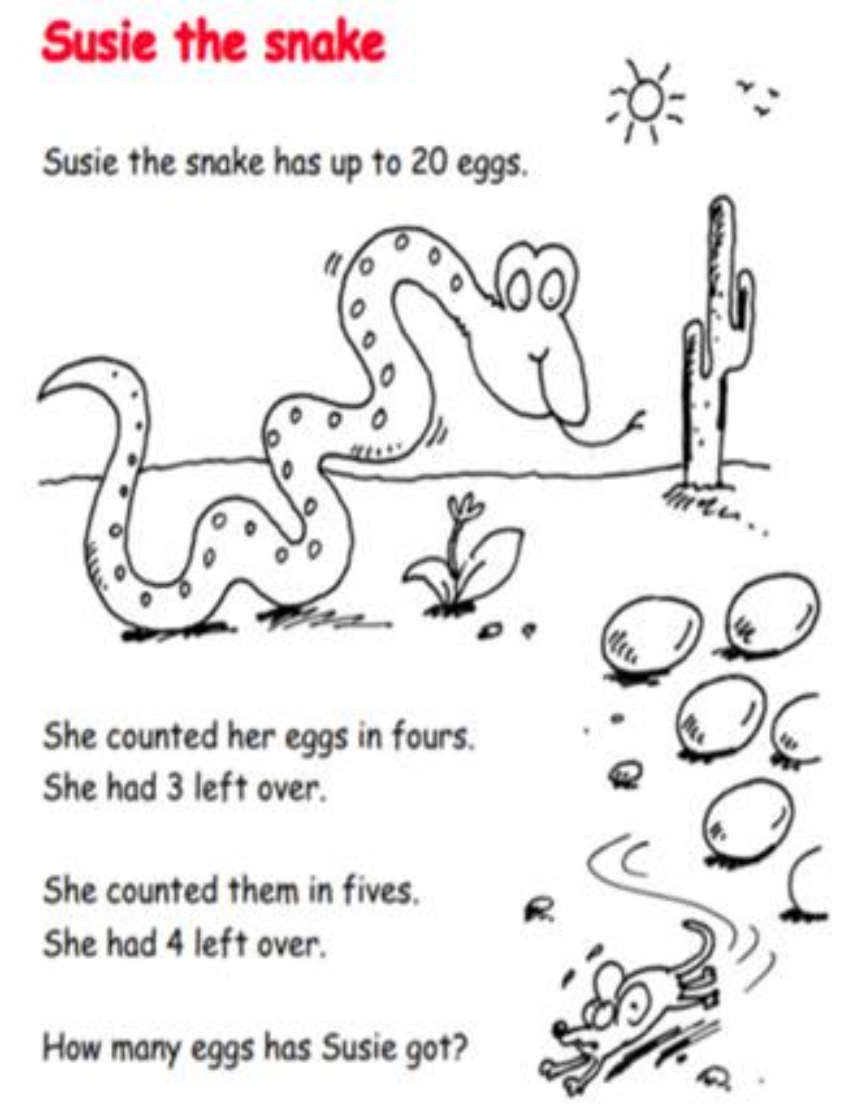
STEM Sentence Examples

Think about colour coding the representation to the STEM sentence.





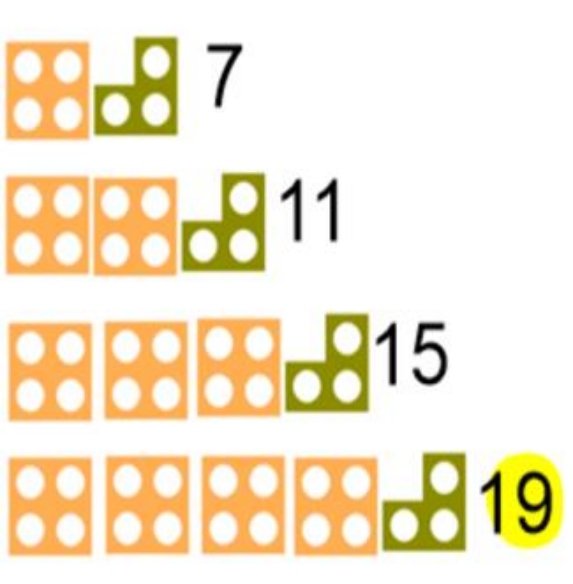
Putting It All Together



Stem sentence-

Discuss with children what is staying the same and what is different. Use the stem sentence.

___ fours is ___ and 3 more is ___



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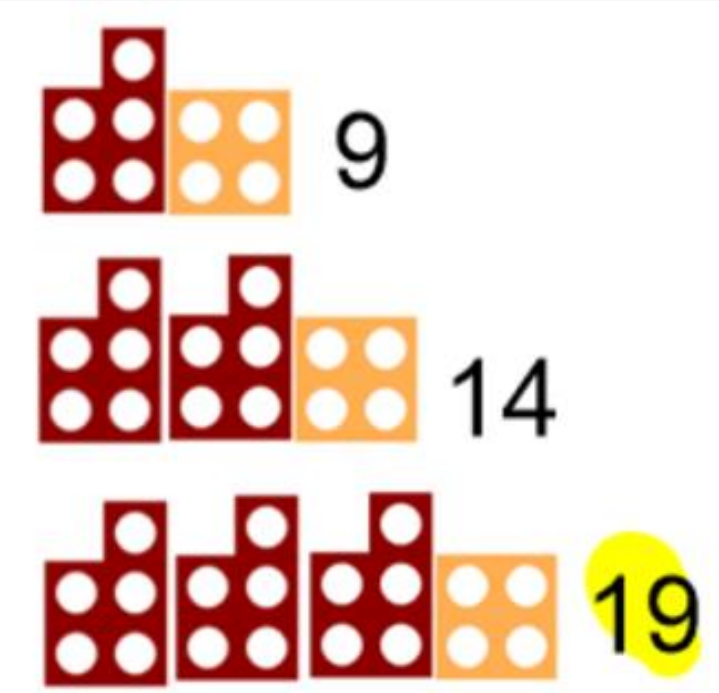
This can be replicated for counting in 5’s and 4 left over.

___ fives is ___ and 4 more is ___



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- Click on ‘My CPD Online’ and ‘Events Due Attend & Unauthorised Bookings’. From here you will be able to locate this event and then click on 'Enter Evaluation’.
- Alternatively please log into your account, using the e-mail link which you will receive from CPD online as soon as the register of attendance has been processed.
- You will be able to download your certificate of attendance once you have completed the evaluation **and** the signed register has been processed by the Training & Development Administration team.
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